M345N10 Second Project – Laplace's equation in an Annulus

This project counts for 40% of the entire module. It was released 20th Feb and it is due in by 23:59 on Monday 13th March. It should be submitted electronically on Blackboard.

The temperature field $u(r, \theta)$ surrounding a pipe obeys Laplace's equation

$$0 = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{in} \quad 1 < r < b, \quad 0 < \theta < 2\pi.$$

On $r = 1$, the temperature is known, while the boundary $r = b$ is insulating,

$$u(1, \theta) = f(\theta), \qquad \frac{\partial u}{\partial r}(b, \theta) = 0.$$

In the θ-direction, u must be 2π-periodic.

This project aims to solve this problem using iterative methods on a regular $M \times N$ polar grid of size $(\delta r, \delta\theta)$ where $M\delta r = b - 1$ and $N\delta\theta = 2\pi$. The problem is a little harder than the 2-D Laplacian considered in lectures, because the coefficients vary with r, as does the grid spacing. We want to discover whether similar results hold in this geometry.

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(1) Write a program to find $u(r, \theta)$ using Jacobi-style iteration. You should take care to implement the three different types of boundary conditions correctly, and make sure your array indices for r and θ are correct. Test the accuracy of your code using trial solutions proportional to $\sin p\theta$ for some integer p, perhaps introducing an artificial forcing term S in the PDE. What ratio M/N seems best to you?

(2) Now modify your code to form an over-relaxation scheme, using Gauss-Seidel. For both Jacobi and the relaxed Gauss-Seidel calculate the number of iterations necessary for the maximum residual to fall below some fixed value. In 2-D, we found a dramatic speed-up of convergence for the right relaxation parameter ω, and this value varied with M and N. Is the behaviour in an annulus similar? Obtain estimates for the optimum ω for some grids. Is this optimum different if the G-S scans first in r or first in θ, or is it the same?

(3) Your aim now is to write a multigrid solver in an annulus using V-cycles. For the Interpolation operator, you should use a weighted average of the immediate neighbours consistent with the polar approximation to the Laplacian at the new point. The Restriction operator can then be the transpose of this matrix. Once again, it is probably better to use G-S WITHOUT over-relaxation, as we hope it has similar error-smoothing properties.

Compare the speed of your multigrid routine with that of the other methods.

(4) Finally, solve as accurately as you can the given problem for the two cases

$$\begin{aligned} &(a) \qquad f(\theta) = \sin\theta \exp(\cos\theta), \\ &(b) \qquad f(\theta) = \sin(3\theta/2), \qquad 0 < \theta < 2\pi. \end{aligned}$$

You may choose the value of b. Use multigrid if possible, but if you cannot get a multigrid routine to work, you may still attempt this part using the iterative methods from part (1) and (2). Is there any reason to anticipate difficulties in either of cases (a) and (b)?