M345N10 First Project – A spinning cylinder in a magnetic field

This project counts for 20% of the entire module. It is due in by 23:59 on Monday 5th February. It should be submitted electronically on Blackboard – instructions will follow.

The magnetic field lines in a plane are contours of the function $\psi(r,\,\theta,\,t)$ where (r,θ) are polar coordinates and t is time. We consider the circular region $0 < r < 2$, which contains conducting fluid which rotates with angular velocity $\omega(r)$, where

$$\omega = \begin{cases} 1 & \text{for} \quad r \leqslant 1 \\ 0 & \text{for} \quad 2 > r > 1. \end{cases} \tag{1}$$

It can be shown that ψ can be written

$$\psi(r,\,\theta,t) = a(r,\,t)\cos\theta - b(r,\,t)\sin\theta \tag{2}$$

where a and b satisfy the linked diffusion equations for a constant parameter η

$$\begin{aligned} a_t - \omega b &= \eta\left[a_{rr} + \frac{a_r}{r} - \frac{a}{r^2}\right] \\ b_t + \omega a &= \eta\left[b_{rr} + \frac{b_r}{r} - \frac{b}{r^2}\right]. \end{aligned} \tag{3}$$

At time $t = 0$, the field lines are parallel with $\psi = r\cos\theta$. Your task is to determine a and b as time increases. You may assume that ψ does not vary in time at $r = 0$ and on $r = 2$.

(1) Write an explicit code to solve this problem, using centred differences in r on a uniform grid with steplengths $(h,\,k)$ in $(r,\,t)$. [You may modify *advdiff.m*, if you wish.]

(2) Discuss the theoretical limitations on k and h for your scheme to work, and illustrate what happens when it begins to fail. In what follows, use values of k which give accurate results

(3) Run your code for the values $\eta = 0.16, 0.04$ & 0.01 and others if you choose, with a steplength $h \leqslant 0.1$. Display contours of ψ at times $t = 3, 6, 9, \infty$ where "∞" means a large enough value that nothing seems to be changing. Also plot $a(t)$ and $b(t)$ for two r-values, one bigger and one less than 1.

(4) Discuss your results. Describe what happens qualitatively, and estimate how the times necessary for things to occur vary with η. If you have time, investigate what happens for other functions $\omega(r)$, for example $\omega = r$ for $r \leqslant 1$, $\omega = 0$ for $r > 1$.

Practical hints: When trying to work out what is going on, it may help to plot the solution every timestep or 2 on the same figure, using the Matlab *pause* command to create a simple animation. Matlab has advice on contouring on polar coordinates, and the routine polarFeb14 I supplied may be of use. If your solution varies on small lengthscales, you may need to reduce h to ensure you resolve the action adequately. And always your k must be small enough to ensure stability.