M345N10 First Project – Advection/Diffusion in an Annulus

This project counts for 10% of the entire module. It is due in by 23:59 on Monday 6th February. It should be submitted electronically on Blackboard – instructions will follow.

The concentration $u(r,\, t)$ of a drug in an annular region surrounding a cylindrical artery is governed by advection and diffusion according to the equation

$$u_t = V(r)u_r + D(u_{rr} + u_r/r) + S(r,\, t) \qquad \text{in } a < r < b \quad \text{and} \quad t > 0, \qquad (*)$$

where r is the radial coordinate. The radial velocity $V(r) = Q/r$, where Q and the diffusivity D are positive constants. The source term $S = 0$, while the boundary conditions are

$$u(a\,,t) = 1 \qquad \text{and} \quad u(b,\, t) = 0 \qquad \text{while} \quad u(r,\, 0) = u_0(r).$$

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(1) Write an explicit code to solve this problem, using centred differences in r on a uniform grid with steplengths $(h,\, k)$ in $(r,\, t)$. [You may modify *advdiff.m*, if you wish.]

(2) Test that your code has errors $O(k,\, h^2)$. To do this, choose values of a, b, Q and D and invent some function $u(r,\, t)$ which satisfies the boundary conditions at $r = a,\, b$. By substituting in $(*)$, work out analytically what S and $u_0(r)$ would have to be for your chosen function to be the solution. Then use your code to find the numerical solution for this artificial forcing S and initial condition u_0. Finally, look at the difference between the exact solution and the numerical solution at some suitable point $(r_0,\, t_0)$, and compare these errors for different grid sizes.

(3) Experiment with your program to find the stability limitations on k and h for your code to be reliable.

(4) After some time, the solution (with $S = 0$, $u_0 = 0$) should settle down to an equilibrium, $u_\infty(r)$. For a fixed ratio b/a, plot on the same diagram your numerical estimates of this equilibrium function for different values of the ratio Q/D, and comment on the results.