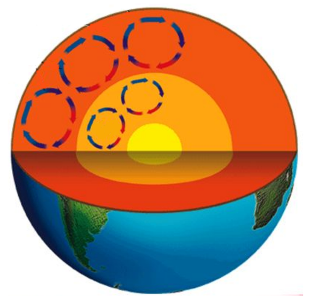

An Annular Model of Convection

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Outline

This project looks at solving the *Convection Equations* in an annulus. If a surface is heated, the heat diffuses throughout the fluid causing the fluid to expand, become more buoyant and rise up. Cooler fluid then moves to replace it and the whole cycle continues with the fluid being advected around. This is called *natural convection* which is driven by gravity and occurs in the center of the Earth and other celestial bodies.
We can model the convection between two spheres in two-dimensions by instead looking at the convection between two circles. This 2-D annular model should capture the key features of the solution. To solve these equations numerically, *Finite Difference Methods* will be used where approximations are made on a grid of points.

The Problem

The governing non-dimensional equations to solve for the temperature T, pressure p and the velocity $\mathbf{u} = (u, v, 0)$ are

$$T_t + \mathbf{u}\cdot\nabla T = \nabla^2 T$$
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = P_r(-\nabla p - R_a T\hat{g} + \nabla^2\mathbf{u})$$
$$\nabla\cdot\mathbf{u} = 0$$
$$\text{in } 1 < r < b \text{ with } T|_{r=1} = 1,\ T|_{r=b} = 0$$

where

- $\hat{g}$ is the direction of gravity here acting in the $-\hat{r}$ direction towards the centre
- P_r is the Prandtl number representing the ratio of the fluid kinematic viscosity to the thermal diffusivity
- R_a is the Rayleigh number measuring the thermal driving force.

We can simplify these equations and eliminate the pressure by taking the curl of the second equation. Instead we now need to solve

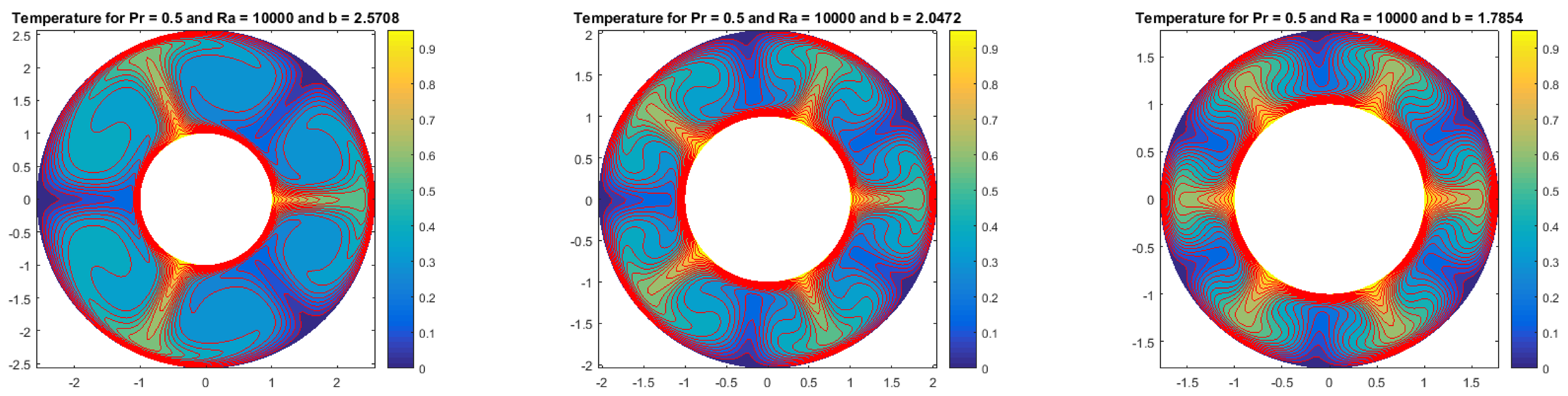
$$\omega_t + \mathbf{u}\cdot\nabla\omega = -P_r R_a \frac{T_\theta}{r} + P_r\nabla^2\omega$$
$$-\omega = \nabla^2\psi$$

where the vorticity ω is defined by $\nabla\times\mathbf{u} = (0, 0, \omega)$ and the stream function ψ by $\mathbf{u} = (\psi_\theta/r, -\psi_r, 0)$.

Natural Convection

There are two different types of solution to the problem - a conductive solution where $\mathbf{u} = 0$ and $T = T(r)$ and a solution where we get convection. Whether we get conduction or what type of convection we get depends on the parameters - the Prandtl and Rayleigh numbers and the size of the annulus. Using 128×256 points on the (r, θ) grid, we can set up the problem by setting the Dirichlet condition on the temperature $T|_{r=1} = 1, T|_{r=b} = 0$ and the initial condition to be zero everywhere else.

In a horizontal plane, a type of natural convection we get if a plate is heated from below (such as heating water in a pan) is Rayleigh-Bénard convection. The number of cells of convection that form depends on the size of the domain. To see if we get something similar in an annular geometry, the annulus size will be set to fractions of the circumference of the inner circle. The equilibrium solutions reached for $P_r = 0.5$, $R_a = 10,000$ and an annulus size of $\pi/2, \pi/3$ and $\pi/4$ are:

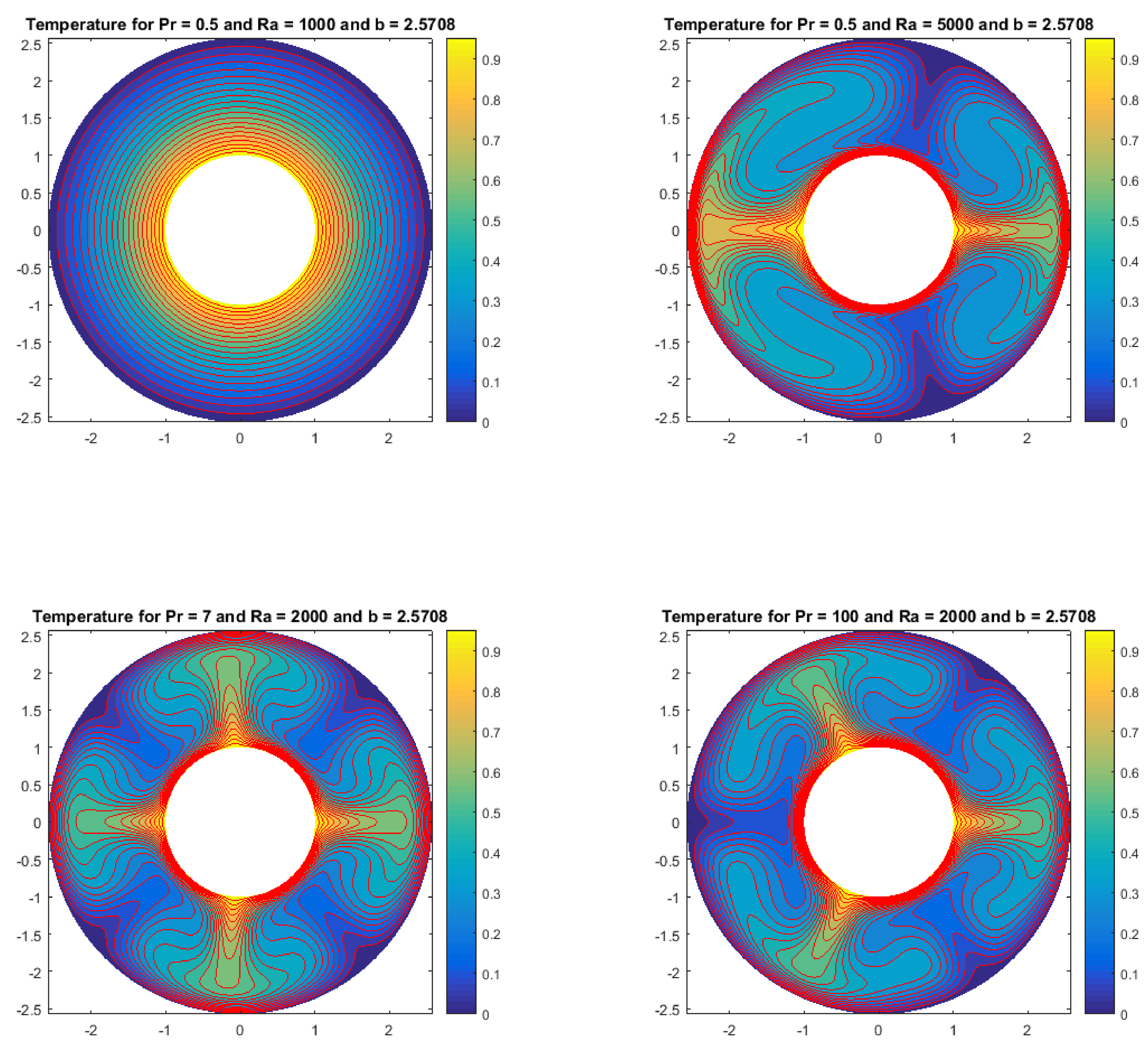


We can clearly see the cells of convection that have been formed. As the annulus size decreases we get more rolls of convection, this is because as $b \to 1$ we approach Rayleigh-Bénard convection in an infinite layer.

Prandtl and Rayleigh Numbers

The convection we observe in the model is caused by small differences and errors made at machine level. These instabilities can take a long time to manifest depending on the parameters. For each Prandtl number, we have a critical Rayleigh number R_c where if we have $R_a < R_c$ we get conduction and if $R_a \geqslant R_c$ these instabilities are amplified and convection develops.

In this specific case for $P_r = 0.5$ we find $R_c \approx 1000$. If P_r is small, diffusion dominates and so we need a larger Rayleigh number to drive the fluid to develop convection. Also, as we increase R_a, the shape of rolls gets sharper and more defined. Now setting $R_a = 2000$ - above the critical value we can now experiment with the Prandtl number. As P_r increases, less diffusion needs to occur before convection develops. For any chosen P_r and R_a, the number of cells varies between 2-4 for the case where $b = \pi/2+1$ and it is possible to have rolls which are not equal in size.



The Method

Using the operator splitting approach we can treat the two processes in the equations separately, first advecting then diffusing.

Advection
We want to advect a quantity Q by $Q_t + \mathbf{u}\cdot\nabla Q = 0$. This can be rewritten as $Q_t + \nabla\cdot(\mathbf{u}Q) = 0$ (since $\nabla\cdot\mathbf{u} = 0$) where $\mathbf{u}Q$ is the *flux*. In polar coordinates this is

$$Q_t + \frac{1}{r}\frac{\partial}{\partial r}(ruQ) + \frac{1}{r}\frac{\partial}{\partial\theta}(vQ) = 0$$

and if we let $F = ruQ$ and $G = vQ$, the equation now becomes $rQ_t + F_r + G_\theta = 0$.
An equation of this form is a *conservation law* so we would like to use a conservative scheme. One such scheme is Lax-Wendroff and in particular the two-step Richtmyer method. Richtmyer is second order in space and time and conserves Q. It involves finding Q at half grid points at half time levels and then using the results to find Q at the actual grid points at integer time levels.

Diffusion
Now we would like to diffuse Q by $Q_t = \kappa\nabla^2 Q + F$ where F is a *forcing* term. It would be best to use an implicit scheme because stable schemes can be produced with larger time steps. Crank-Nicolson has been used here which gives

$$\frac{Q^{j+1}_{m,n} - Q^{j}_{m,n}}{\delta t} = \frac{\kappa}{2}\left(\Delta^2 Q^{j+1}_{m,n} + \Delta^2 Q^{j}_{m,n}\right) + F^{j}_{m,n}$$

where Δ^2 is the finite difference approximation to the polar Laplacian. An iterative scheme such as Gauss-Seidel can be used to find $Q^{j+1}_{m,n}$ from this equation. To improve the speed of the code, a Multigrid method can be used to switch between coarse and fine grids.

Conclusion

For a given annulus size we can find the average number of rolls that develops by experimenting with the parameters. However, it is not entirely clear how this number varies as we change P_r and R_a and this is because these two numbers are linked. Given any fluid, if we know the Prandtl and Rayleigh numbers we should be able to use this model to find the solution.