Nonlinear animal coat patterns: Turing's problem.

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Introduction

The main aim of this project is the research on the topic proposed by Alan Turing on how colour **patterns** form on the **skin** of certain **animals** - e.g. cheetahs, giraffes, etc.

The resolution of this problem ultimately reduces to the numerical simulation of a system of equations of the form:

$$u_t = \gamma(a - u + u^2 v) + (u_{xx} + u_{yy})$$
$$v_t = \gamma(b - u^2 v) + d(v_{xx} + v_{yy})$$

with homogeneous Neumann boundary conditions.

When the parameter choice is *appropriate,* the fixed point $u_0(x,y) = a + b,\, v_0(x,y) = \frac{b}{(a+b)^2}$ becomes unstable. Hence that will be the initial condition for our simulation.

Methods

Finite differences are employed to discretise and solve the system of PDEs. In particular, a Gauss Seidel solver integrated with a **multigrid** method are used to get the solution at each instant. The time evolution is captured via a Crank Nicholson scheme.

The nonlinear source term - i.e. $\gamma(a - u + u^2 v)$ - is appropriately linearised for each of the equations and called $Q(u)$ in the discretisation scheme below:

$$u_{n,m}^{j+1}\left[1 + 2\theta\Delta t d\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) - \theta\Delta t\left(\frac{\partial Q}{\partial u}\right)_{n,m}^{j}\right] = u_{n,m}^{j} + \Delta t\left[Q_{n,m}^{j} - \theta\left(\frac{\partial Q}{\partial u}\right)_{n,m}^{j} u_{n,m}^{j}\right]$$
$$+\Delta t(1-\theta)d\left[\frac{u_{n-1,m}^{j} - 2u_{n,m}^{j} + u_{n+1,m}^{j}}{\Delta x^2} + \frac{u_{n,m-1}^{j} - 2u_{n,m}^{j} + u_{n,m+1}^{j}}{\Delta y^2}\right]$$
$$+\theta\Delta t d\left[\frac{u_{n-1,m}^{j+1} + u_{n+1,m}^{j+1}}{\Delta x^2} + \frac{u_{n,m-1}^{j+1} + u_{n,m+1}^{j+1}}{\Delta x^2}\right]$$

where *n* and *m* represent the grid point in the *x*- and *y*-axis, respectively; and *j* denoted the time point. Note that $\theta = 0.5$ (Crank Nicholson).

Special attention was put into the resolution on **boundaries** and corners, where the Neumann boundary conditions were imposed as

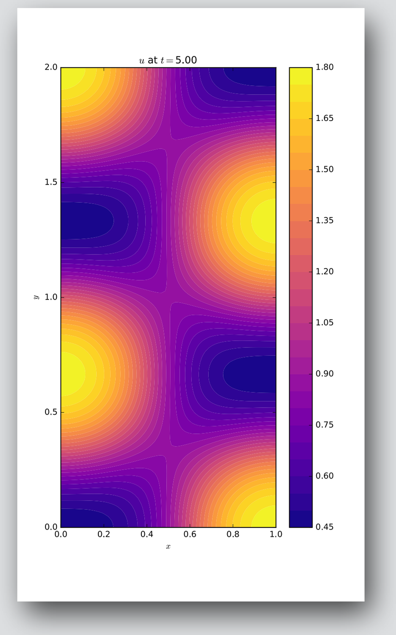
$$\frac{\partial u}{\partial x}(x = 0, y) = \Phi_{x=0}(y) \quad \Rightarrow \quad \frac{u_{-1,m} - u_{1,m}}{2\Delta x} = \Phi_{x=0}(y_m)$$

Spotty Pattern

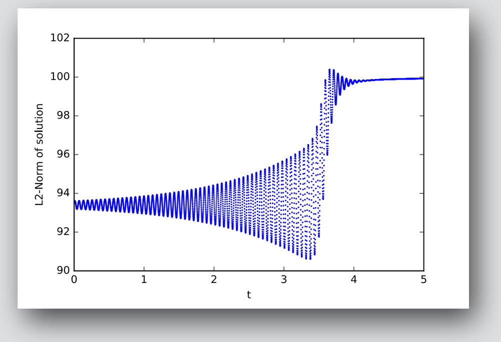
When the shape of the domain is *square enough* spotty patterns arise. This can be linked to what happens on the **body** of the cheetah.

Values of $\gamma = 100,\ a = 0.07,\ b = 0.95,\ d = 10$ are used for all the showed simulations.

In order to check whether the obtained solution is a **new equilibrium** state, the L2-norm of the solution is computed at each instant of the simulation and plotted. This result is displayed at the bottom right of this grey block.



If the 1xL rectangle is stretched, by increasing the value of L, the patterns that form are made of stripes. For values of L = 1, 2, 3, 4 and 5, spots appear at the new equilibrium. For values of L = 6 and larger, stripes arise.



Stripy Pattern

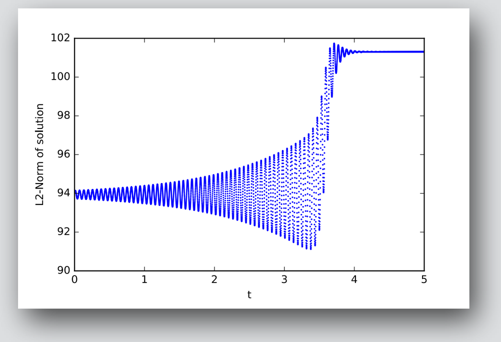
When the shape of the domain is *slender enough* stripy patterns occur. This can be linked to what happens on the **tail** of the cheetah.

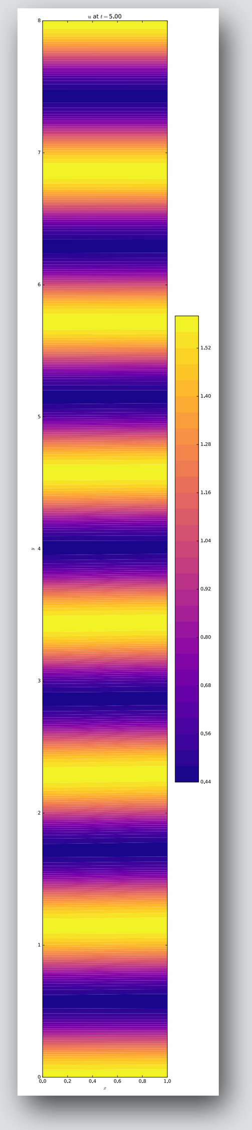
The results displayed on the right correspond to a rectangle with L = 8.

Stripes start to form around the central area of the domain, becoming then clearer on the rest of the rectangle.

As in the previous case, the solution is a new **equilibrium** state. This can be checked by taking the norm of the solution at each instant and observing its evolution.

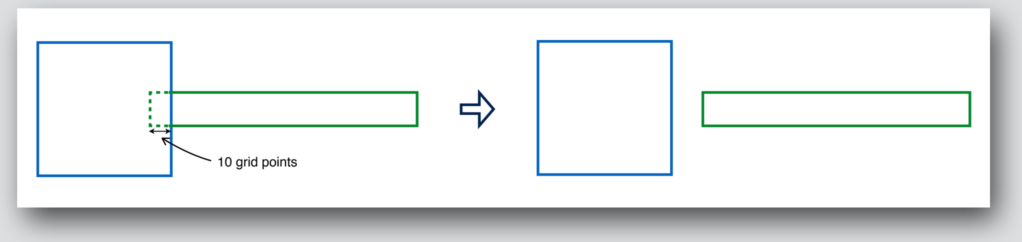
This can be seen in the picture on the right.





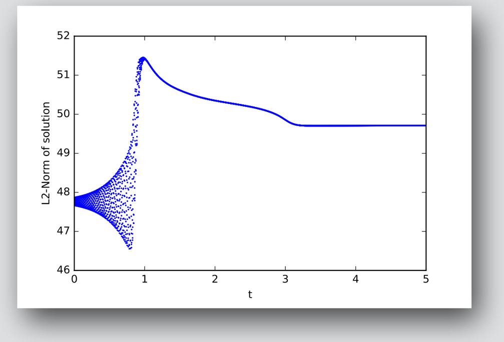
Mixed Pattern

The last simulation is carried out for an overlapping domain. This domain comprises a 2x2 square - which represents the body - and a 4x0.5 rectangle representing the tail.



The tail rectangle has a small region which overlaps with the body square. The system is firstly solved for the body. Then, the interim **solution is used to obtain** the appropriate Neumann **boundary conditions** on the overlapping region of the tail. Subsequently, the system is solved for the tail and the boundary conditions for the body are computed. This cycle is repeated four times at each instant.

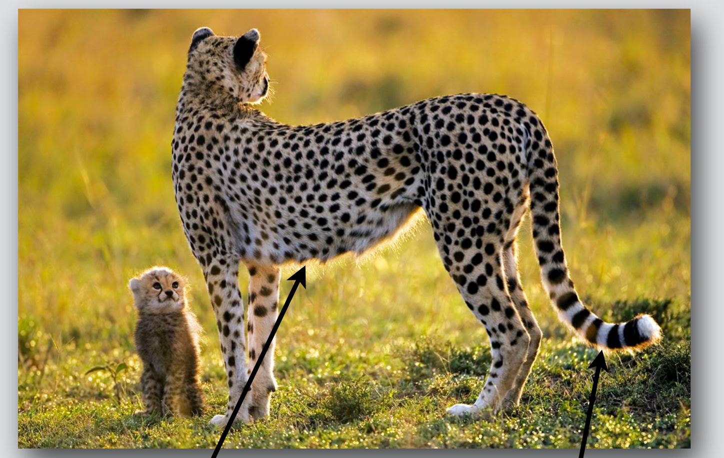
The approach to the new equilibrium state is somewhat different to the elementary cases. The system first gets stabilised, and then it moves towards the final state.



The results displayed on the right show that spots are indeed formed on the square part of the domain, whilst stripes arise on the rectangular region, perpendicular to the longer dimension.

One can visually verify that the **boundary conditions hold** at the body-tail **linkage**. In fact, the overall pattern is not as perfectly symmetric as it was in the elementary cases. **Asymmetries** and untidiness are induced by the connection of two domains which have an influence on each other.

As suggestions for **future work**, we would suggest the further exploration of the **transition** point between the formation of spots and stripes. It would also be interesting to simulate this problem on a **tapered** domain.



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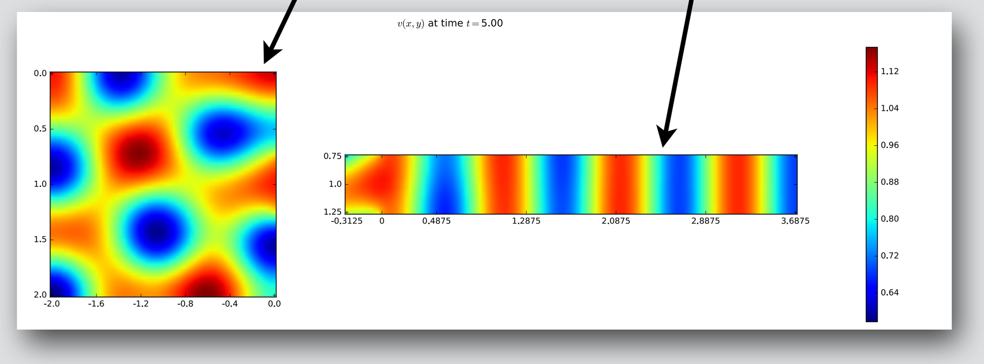

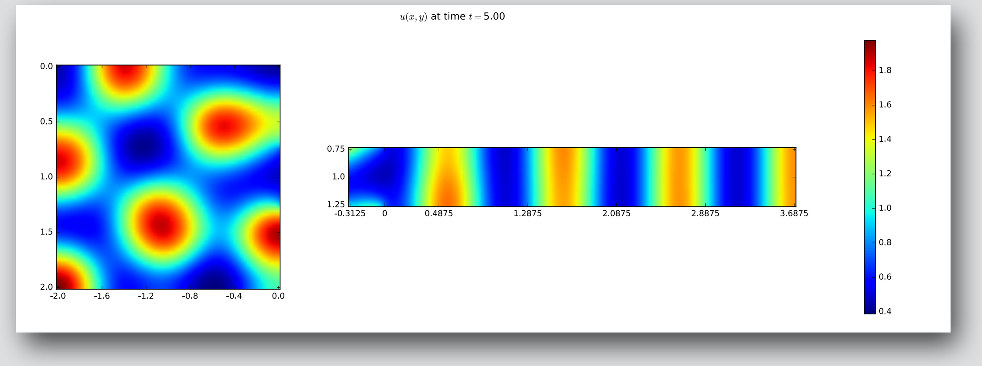

Videos of pattern formation can be found at:
Spots: https://youtu.be/AY3D_EC3ODo
Stripes: https://youtu.be/6vZIBEIwZ2w
Mixed: https://youtu.be/i2ZyX25NB2E

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References:

[1] Unknown, (2018), Cheetah mom [ONLINE]. Available at: http://rebrn.com/re/baby-cheetah-with-his-mom-876280/