Name (IN CAPITAL LETTERS!): ....

CID: ....

Question 3.

(a) Express in both standard and polar form the complex number

$$\frac{1+i}{e^{i\pi/3}}.$$

Deduce that

$$\cos\tfrac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(b) Assuming that the formula for $\sin(A+B)$ holds even when A and B are complex, show that the only complex numbers $z = x + iy$ such that $\sin(x + iy) = 0$ are real.

[You may also assume $\cosh[t] = \cos[it]$, $i \sinh[t] = \sin[it]$ and standard properties of these functions.]

(a) We have $(1 + i) = \sqrt{2}\exp(\frac{1}{4}i\pi)$ and $\exp(-i\pi/3) = \frac{1}{2} - \frac{1}{2}i\sqrt{3}$. Therefore

$$(1+i)e^{-i\pi/3} = \sqrt{2}e^{-i\pi/12} = \tfrac{1}{2}(1+i)(1-i\sqrt{3}) = \frac{1+\sqrt{3}}{2} + i\frac{1-\sqrt{3}}{2} \qquad [\mathbf{2}+\mathbf{2}]$$

Taking the real part, we have

$$\sqrt{2}\cos\tfrac{1}{12}\pi = \frac{1+\sqrt{3}}{2} \qquad \text{or} \qquad \cos\tfrac{1}{12}\pi = \frac{1+\sqrt{3}}{2\sqrt{2}} \qquad [\mathbf{1}]$$

(b) We have

$$\sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y$$

If $\sin z = 0$, both the real and imaginary parts must be zero, so that $\sin x \cosh y = 0 = \cos x \sinh y$. But $\cosh y \geqslant 1$, so we must have $\sin x = 0$. This means $\cos x \neq 0$, and so we must also have $\sinh y = 0$. But this only happens when $y = 0$, so that z must be real. **[5]**