Name (IN CAPITAL LETTERS!): ....

CID: ....

Question 1.
A function $A(x)$, satisfies the equation

$$A'' = x^2 A \qquad \text{with} \quad A(0) = 1, \qquad \text{and} \quad A'(0) = 0.$$

Deduce the values of $A''(0)$ and $A'''(0)$.
Differentiate the equation n times using Leibniz' formula. Hence find a power series expansion of $A(x)$ about the point $x = 0$, giving terms up to and including x^8.

Solution: Differentiating once, we have $A''' = 2xA + x^2 A'$. Substituting $x = 0$, we have $A'''(0) = 0$. Also $A''(0) = 0$ by substitution. **[1]**

By Leibniz

$$A^{(n+2)} = x^2 A^{(n)} + n(2x)A^{(n-1)} + \tfrac{1}{2}n(n-1)(2)A^{(n-2)}. \qquad \textbf{[2]}$$

Substituting $x = 0$, we have

$$A^{(n+2)}(0) = n(n-1)A^{(n-2)}(0). \qquad \textbf{[2]}$$

This relation tells us that the $(n+4)$'th derivative is proportional to the nth derivative. As the 1st, 2nd and 3rd derivatives are all zero, it follows that the only non-zero derivatives are for $n = 4k$ for $k = 0, 1\ldots$ We have $A(0) = 1$ and so $A^{(4)}(0) = 2$ and $A^{(8)}(0) = (6)(5)(2) = 60$.

Thus the Maclaurin series is

$$A(x) = A(0) + \frac{x^4}{4!}A^{(4)}(0) + \frac{x^8}{8!}A^{(8)}(0) + O(x^{12}) = 1 + \tfrac{1}{12}x^4 + \frac{x^8}{(8)(7)(4)(3)} + O(x^{12}). \quad \textbf{[5]}$$