Progress Test, Week 6, Question 4:

It is not generally known that Prof Thomas once had a beard. It started growing at $t = 0$, where t denotes time measured in suitable units. The length, L, of his beard in centimetres is given at time t by

$$L = \begin{cases} 0 & \text{for } t \leqslant 0 \\ \dfrac{t^2}{1+4t^4} & \text{for } 0 < t < 1, \\ \frac{1}{5}\exp(1-t) & \text{for } t \geqslant 1 \end{cases}$$

Justifying your answers appropriately,

(a) For which values of t is the function $L(t)$ continuous?

(b) For which values of t is the function $L(t)$ differentiable?

(c) What is the longest his beard has ever been? You may assume this occurred during $0 < t < 1$.

Solution: (a) $L(t)$ is clearly continuous for $t < 0$, for $0 < t < 1$ and for $t > 1$, but we should check the behaviour at $t = 0$ and $t = 1$, where respectively the beard started growing and perhaps Prof Thomas started shaving.

$$\lim_{t\to 0}\left[\frac{t^2}{1+4t^4}\right] = 0 \qquad \text{and} \qquad \lim_{t\to 1}\left[\frac{t^2}{1+4t^4}\right] = \tfrac{1}{5} = \lim_{t\to 1}\left[\tfrac{1}{5}\exp(1-t)\right].$$

Thus $L(t)$ is continuous also at $t = 0$ and at $t = 1$. $L(t)$ is continuous for all t. **[2]**

(b) $L(t)$ is clearly differentiable except possibly at $t = 0$ and $t = 1$. Now for $t < 0$ $L'(t) = 0$, and for $0 < t < 1$,

$$L'(t) = \frac{2t(1+4t^4) - t^2(16t^3)}{(1+4t^4)^2} = \frac{2t(1-4t^4)}{(1+4t^4)^2}.$$

As $t \to 0$ this tends to 0, in agreement with the behaviour for $t < 0$, so $L(t)$ is differentiable at $t = 0$. As $t \to 1$ from below, $L'(t) \to -\frac{6}{25}$.

Now for $t > 1$, $L'(t) = -\frac{1}{5}\exp(1-t) \to -\frac{1}{5}$ as $t \to 1$. As these values are different ($-\frac{1}{5} \neq -\frac{6}{25}$), $L(t)$ is not differentiable at $t = 1$ – the gradient changes instantaneously. We conclude that $L(t)$ is differentiable for all $t \neq 1$. **[3]**

(c) We note that $L'(t) = 0$ at $t = 0$ and at $t = 1/\sqrt{2}$. $L' \geqslant 0$ for $0 < t < 1/\sqrt{2}$, and $L' \leqslant 0$ for $t > 1/\sqrt{2}$. Thus $L(t)$ achieves its maximum value at $t = 1/\sqrt{2}$. This maximum is

$$L(1/\sqrt{2}) = \frac{1/2}{1+1} = \tfrac{1}{4}. \qquad \textbf{[5]}$$

So once his beard was 0.25cm long, if this question can be believed.

[**Note to markers:** They should give some justification to parts (a) and (b) but it need not be exactly as I've done it. Some will calculate the 2nd derivative of L. As ever, use your judgement.]