Progress Test, Week 5, Question 3:

Using only the methods covered so far in the course, find the following two limits:

$$(a)\quad \lim_{x\to 0}\left[\frac{\exp(-x)-1+\sin x}{1-\cos x}\right] \qquad (b)\quad \lim_{x\to 2}\left[\frac{\log(x-1)}{\sin(\pi x)}\right].$$

Solution:

(a) We have $\exp(-x) = 1-x+\frac{1}{2}x^2+O(x^3)$, $\sin x = x+O(x^3)$ and $\cos x = 1-\frac{1}{2}x^2+O(x^4)$. So

$$\lim_{x\to 0}\left[\frac{\exp(-x)-1+\sin x}{1-\cos x}\right] = \lim_{x\to 0}\left[\frac{1-x+\frac{1}{2}x^2-1+x+O(x^3)}{1-(1-\frac{1}{2}x^2+O(x^4))}\right] = \lim_{x\to 0}\left[\frac{\frac{1}{2}x^2+O(x^3)}{\frac{1}{2}x^2+O(x^4)}\right] = 1$$

[**5**]

(b) Write $x = 2+y$. Then $\log(x-1) = \log(1+y) = y-\frac{1}{2}y^2+O(y^3)$ and we also have $\sin\pi x = \sin(2\pi+\pi y) = \sin\pi y = \pi y + O(y^3)$. Thus

$$\lim_{x\to 2}\left[\frac{\log(x-1)}{\sin(\pi x)}\right] = \lim_{y\to 0}\left[\frac{y+O(y^2)}{\pi y+O(y^3)}\right] = \frac{1}{\pi}.$$ [**5**]

Note to markers: If they use de l'Hôpital's rule, deduct two marks even if they get the right answers, as they're instructed not to. Only exact this penalty once.