Progress test week 4, Question 2:

Write down the power series for $\sin x$ and $(1+x)^{-1/2}$ giving terms up to and including x^3. Hence express as a power series in x the function

$$\frac{1}{\sqrt{1+\sin x}}$$

including all terms up to and including x^3.

Make an intelligent guess as to for which values of x the infinite series conv erges.

Solution:

$$\sin x = x - \tfrac{1}{6}x^3 + \dots \qquad [\mathbf{1}]$$

$$(1+x)^{-1/2} \simeq 1 - \tfrac{1}{2}x + (-\tfrac{1}{2})(-\tfrac{3}{2})(\tfrac{1}{2})x^2 + (-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})(\tfrac{1}{6})x^3 = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 \quad [\mathbf{2}]$$

provided $|x| < 1$.

$$\begin{aligned}(1+\sin x)^{-1/2} &= 1 - \tfrac{1}{2}\sin x + \tfrac{3}{8}\sin^2 x - \tfrac{5}{16}\sin^3 x + \dots \\ &= 1 - \tfrac{1}{2}(x - \tfrac{1}{6}x^3 + \dots) + \tfrac{3}{8}(x - \tfrac{1}{6}x^3 + \dots)^2 - \tfrac{5}{16}(x + \dots)^3 + \dots \\ &= 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^3 + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \dots \\ &= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{11}{48}x^3 + O(x^4).\end{aligned} \qquad [\mathbf{5}]$$

We need $|\sin x| < 1$ so $|x| < \tfrac{1}{2}\pi$ should be ok. When $x = -\tfrac{1}{2}\pi$ we know the function is infinite, so we might guess that $|x| < \tfrac{1}{2}\pi$ is necessary and sufficient for convergence. [**2**]

The guess $x = \tfrac{3}{2}\pi$ is also intelligent and should be allowed – other guesses at the marker's discretion. Note that the above is some way from being truly rigorous.

Total : [**10**]

[**Note to marker:** Some may use a Maclaurin or Taylor series. If they get it right, nevertheless deduct one mark for not obeying the 'Hence' instruction in the question.]