M1M1: Sheet 6: Mean Value Theorem and Taylor's Theorem

1. Consider the function $f(x) = x^3 - 8x - 5$ in the domain $-1 \le x \le 4$. Find a number c, with $-1 < c < 4$, such that

$$\frac{f(4) - f(-1)}{5} = f'(c).$$

Now let $f(x) = \frac{4}{x}$. Show that there is no number c with $-1 < c < 4$ such that

$$\frac{f(4) - f(-1)}{5} = f'(c).$$

Why does this not contradict the mean value theorem?

2. Apply the mean value theorem to $\tan^{-1}(x)$ to show that

$$\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$$

where $0 < a < b$. Hence obtain the value of $\tan^{-1}(21/20)$ to 2 decimal places of accuracy.

3. Use the mean value theorem to show that

$$\frac{1}{\sqrt{66}} < \sqrt{66} - 8 < \frac{1}{8}.$$

4. Use the first few terms of an appropriate series expansion to estimate the value of $8.1^{1/3}$, giving your answer correct to four decimal places.

5. Find the first few terms in the Taylor series of $\sin(x + a)$ about $x = 0$ and hence verify the identity

$$\sin(x + a) = \sin x \cos a + \sin a \cos x$$

6. Let

$$y(x) = \sin(m \sin^{-1}(x))$$

where m is some real number. Show that

$$(1 - x^2)y'' - xy' + m^2 y = 0$$

By differentiating this equation n times, show that

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^n y}{dx^n} = 0.$$

Now set $x = 0$ and hence derive the following Taylor expansion about $x = 0$:

$$y(x) = mx + m(1-m^2)\frac{x^3}{3!} + m(1-m^2)(9-m^2)\frac{x^5}{5!} + \dots$$

Show that this series converges for $|x| < 1$.

7. Show that $y(x) = \tan(x)$ satisfies the equation

$$\frac{dy}{dx} = 1 + y^2.$$

By repeated differentiation of this equation, find the higher derivatives of $y(x)$ and hence determine the first three non-zero terms of the Taylor expansion of $\tan x$ about $x = 0$. Check your answer by using the series for $\sin x / \cos x$.

8. Derive the Taylor series about $x = 0$ for the function

$$\log\left(\frac{1+x}{1-x}\right).$$

State its radius of convergence and use the series to obtain the value of $\log(5/3)$ to 4 decimal places of accuracy.

9. Let $f(x)$ be differentiable in the neighbourhood of $x = a$ as many times as we like. Use an infinite Taylor series to show that, when h is small,

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} + E_1$$

where the error

$$E_1 = -\frac{h^2 f'''(a)}{6} + O(h^4).$$

Show also that

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} + E_2$$

where the error

$$E_2 = -\frac{h^2 f''''(a)}{12} + O(h^4).$$

Now let $f(x) = \sin x$ and $h = \pi/12$. From the above approximations, find the values of $f'(\pi/4)$ and $f''(\pi/4)$ and compare with the exact values.