M1M1: Problem Sheet 4: Differentiation

1. Find the derivatives of the following function *from first principles* (i.e., use the definition of the derivative and algebraic manipulation; strictly, you should not use Taylor series or the binomial series for non-integers):

$$(a)\ x^3; \quad (b)\ x^{1/3}; \quad (c)\ \sqrt{x^2-1}; \quad (d)\ \cos x; \quad (e)\ \tan x; \quad (f)\ \sin\sqrt{x}.$$

2. Using any rules of differentiation that you like (e.g. the product rule, quotient rule, chain rule), find the derivatives of the following functions:

(a) $\sin(x^2)$; (b) $\sin^2 x$; (c) $\sin(2x)$; (d) 10^x; (e) $(\sin x)^x$; (f) $\tan(\sin x)$;

(g) $\cot^{-1}(x)$; (h) $\exp(3x^2+5x+2)$; (i) $\exp(-x)\cosh(2x)$; (j) $\log(x)\exp(x)$;

$$(k)\ \sin^{-1}(x); \quad (l)\ \sec x; \quad (m)\ \log|\sec x + \tan x|; \quad (n)\ (x^2-1)^{1/2};$$

$$(o)\ \frac{x}{(x^2-1)^{1/2}}; \quad (p)\ \frac{1}{(x^2-1)^{3/2}}; \quad (q)\ \log(x)\sin^{-1}(x); \quad (r)\ \tan^{-1}(x).$$

3. A curve in the (x, y)-plane is given in polar coordinates (r, θ) (where $x = r\cos\theta,\ y = r\sin\theta$) by the equation $r(\theta) = \sin\theta$. Show that

$$\frac{dy}{dx} = \tan 2\theta.$$

Another curve is given by the equation $r(\theta) = 1+\sin^2\theta$. Find $\frac{dy}{dx}$ at the point corresponding to $\theta = \pi/4$.

4. Find the location and nature of the stationary points of the following curves:

$$\begin{aligned}
&(a) \quad y = 2x^3 + 15x^2 - 84x;\\
&(b) \quad y = x^5 - 5x + 1;\\
&(c) \quad y = \log 2x + \frac{1}{x}.
\end{aligned}$$

5. If

$$x(s) = \cos 2s, \quad y(s) = s - \tan s,$$

show that

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{(1+x)^3}}.$$

6. Given that $xy(x+y)^a = b$, where a and b are constants, show that

$$\frac{dy}{dx} = -\frac{y}{x}\left(\frac{(a+1)x+y}{(a+1)y+x}\right).$$

7. Establish the relation

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}.$$

Verify this expression for the example $y = (1+x)^{-1}$.

8. If

$$y = \left[x + 1 + \sqrt{x^2 + 2x + 2}\right]^p,$$

show that

$$\sqrt{x^2 + 2x + 2}\frac{dy}{dx} = py.$$

Differentiate this equation to show that

$$(x^2 + 2x + 2)\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} - p^2y = 0.$$

Now use Leibniz's formula to show that, at $x = 0$,

$$2\frac{d^{n+2}y}{dx^{n+2}} + (2n+1)\frac{d^{n+1}y}{dx^{n+1}} + (n^2 - p^2)\frac{d^ny}{dx^n} = 0.$$

9. Use Leibniz's formula to show that:

$$\frac{d^n}{dx^n}\left[(1+x^2)\exp(x)\right] = 1 + n(n-1) \quad \text{when } x = 0.$$

10. Find $f^{(n)}(x)$ if

$$f(x) = \frac{1}{x^2 + 3x + 2}.$$

11. A sheet of metal of fixed area A is to be made into a right circular cylinder with closed ends. Show that the volume of the cylinder is a maximum when its length is equal to its diameter and find this maximum volume.

12. Prof Liegroup keeps a clue to the Treasure Hunt on the top of a high shelf of his office, only accessible with a ladder. To reach his office, you have to walk down two passages, of widths a and b, which meet at a right-angled corner. Show that the longest ladder that can be carried horizontally around this corner is of length $(a^{2/3} + b^{2/3})^{3/2}$.