M1M1: Problem Sheet 3: Convergence of Power Series and Limits

1. Use the Ratio Test to find the Radii of Convergence of the power series for

$$(a)\quad \cos(x) \qquad (b)\quad \log(1+x) \qquad (c)\quad (1+x)^{\alpha}$$

where α is not a positive integer.

2. Consider the two functions

$$f(x) = \frac{1}{2-\cosh x} \qquad g(x) = \frac{1}{1+\exp(x)}.$$

(a) Explain why it is to be expected that the Radius of Convergence of the Maclaurin series for $f(x)$ is $\log(2+\sqrt{3})$.
(b) It is found using a computer that the power series for $g(x)$ appears to have a radius of convergence R where $3.1 < R < 3.2$. Can you think of a reason why this might be? [Hint – think of x as a complex number].

3. Evaluate the following limits:

$$(a) \lim_{x\to 1} \frac{x^3-1}{x}; \quad (b) \lim_{x\to -2} \frac{x^2+5x+6}{x^2+3x+2}; \quad (c) \lim_{x\to\infty} \frac{x^5+7x^3}{4x^5+x^2}$$

$$(d) \lim_{x\to 1} \frac{x^3-1}{x-1}; \quad (e) \lim_{x\to\infty} \frac{(1+x^2)^{1/2}}{x}; \quad (f) \lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x};$$

$$(g) \lim_{x\to 0} \frac{\tan x}{x}; \quad (h) \lim_{x\to 1} \frac{\sin(x-1)}{x^2-5x+4}; \quad (i) \lim_{x\to 2} \frac{\tan(p(x-2))}{\tan(q(x-2))}$$

4. Use the result that $\lim_{x\to\infty} xe^{-\alpha x} = 0$ for $\alpha > 0$ to show that

$$\lim_{t\to 0^+} t^{\alpha} \log t = 0$$

where the notation $t \to 0^+$ means that t tends to zero through positive values.

5. Evaluate the following limits:

$$(a) \lim_{x\to 0} \frac{x+\sin x}{x+x^2}; \quad (b) \lim_{x\to 1} \frac{\log x}{x^2-1}; \quad (c) \lim_{x\to\pi/2} \frac{1-\sin x}{(x-\pi/2)^2};$$

$$(d) \lim_{x\to\pi/2} (\sec x - \tan x); \quad (e) \lim_{x\to 0} \left[(\sec x)^{x^{-2}}\right]; \quad (f) \lim_{x\to\infty} \left[x^{1/3}\left((x+1)^{2/3} - x^{2/3}\right)\right];$$

$$(g) \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right); \quad (h) \lim_{x\to\infty} \left(1+\frac{c}{x}\right)^x; \quad (i) \lim_{x\to 1} \frac{x^2-2x+1}{x^4+x^3-7x+5}$$

6. For each positive integer n, we define a function

$$f_n(x) = \begin{cases} n & \text{if} \quad 1/n < x < 2/n \\ 0 & \text{otherwise} \end{cases}$$

What is the maximum value of $f_n(x)$, and what is its integral over all x? Show that for each value of x, $\lim_{n\to\infty} f_n(x) = 0$. Deduce that

$$\lim_{n\to\infty} \left(\max_x [f_n(x)]\right) \neq \max_x \left[\lim_{n\to\infty} (f_n(x))\right] \quad \text{and} \quad \int_{-\infty}^{\infty} \lim_{n\to\infty} [f_n(x)]\, dx \neq \lim_{n\to\infty} \left[\int_{-\infty}^{\infty} f_n(x)\, dx\right]$$