M1M1: Problem Sheet 2: Power series expansions

1. In lectures (http://www.ma.ic.ac.uk/~ajm8/M1M1/cozzin.pdf)
we proved that for cos and sin defined by infinite series,

(a) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

(b) there is a number τ with $1.4 < \tau < 1.6$ such that $\cos\tau = 0$.

Using only these results and others derived from the infinite series, show that

(i) $\cos^2\theta + \sin^2\theta = 1$, (ii) $\cos(\theta+\tau) = -\sin\theta$ (iii) $\cos(2\tau) = -1$

(iv) $\cos 4\tau = 1$ (v) $\cos(\theta + 4\tau) = \cos\theta$ for all θ.

2. The hyperbolic sine and cosine, denoted sinh x and cosh x, are respectively the odd and even parts of the exponential function $\exp(x)$. Use this fact to write down the series expansions for sinh x and cosh x.

The hyperbolic tangent, tanh x, is defined as

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

Use the series expansions for sinh and cosh to find the first 3 non-zero terms in the series expansion of tanh x.

3. Derive the following expression for the inverse hyperbolic tangent

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Use this expression to find the series expansion of $\tanh^{-1}x$.

4. Using well-known series expansions, but without using Maclaurin ideas, find the first three non-zero terms in the power series of:

(a) $(1+x)\exp(x)$; (b) $\sin(x+1)$; (c) $\exp(x)\log(1+x)$;

(d) $\dfrac{1}{2-\exp(x)}$; (e) $\sec x$; (f) $\tan x$;

(g) $\log(1+\exp(x))$; (h) $\cos(\sin(x))$.

[Remember – only expand in variables which are small as $x \to 0$]

5. Find the Maclaurin series for the function

$$f(x) = \frac{1}{x^2+3x+2}.$$

[Hint - how can you render the repeated differentiation easier to perform?]

6. Find the first three non-zero terms of the Maclaurin series of the functions

(a) $\exp(x)\cos x$; (b) $\tan^{-1}(x)$; (c) $\sec x$.

Have you any ideas how you might find ALL terms in the series for (a) and (b) more easily?

Check that your result for (c) agrees with Q4.