M1M1: Problem Sheet 1: Functions

1. Solve the following equations:

$$(a) \quad x^2 - 4x - 4 = 0 \qquad\qquad (b) \quad 5x^3 - 8x^2 + 3x = 0$$

$$(c) \quad \frac{2u+3}{u-1} = \frac{3u-2}{5u-21} \qquad\qquad (d) \quad 3 = \tfrac{1}{2}\left(e^v + e^{-v}\right).$$

2.(a) If

$$f(x) = \frac{x+1}{x-1} \quad \text{and} \quad g(x) = \frac{x-1}{x+1},$$

(except at the singular points) find expressions for $f(g(x)), g(f(x)), f(f(x))$ and $g(g(x))$. Give some thought to the points where f and g are "infinite."
(b) Find the function inverse to $f(x) = \frac{x-2}{x-1}$.
What does your result tell you about the graph $y = f(x)$?

3. More generally, consider functions $f(x)$ of the form

$$f(x) = \frac{ax+b}{cx+d} \qquad \text{for} \quad x \neq -\frac{d}{c}, \qquad (*)$$

where a, b, c and d are real constants with $c \neq 0$.

(a) Find the inverse function $f^{-1}(x)$, assuming it exists. By computing the composed function $f^{-1}(f(x))$, show that the inverse exists provided $ad \neq bc$. What happens in the special case $ad = bc$?

(b) Find the relation between a, b, c and d for which $f^{-1}(x) \equiv f(x)$.

(c) Define a new function $g(x) = f(x - e)$ where $e = (a + d)/c$. Use the result of part (b) [remember it applies to **ALL** functions of the form (*)] to show that $g^{-1}(x) \equiv g(x)$.

4. The *Heaviside step function* $H(x)$ is defined as follows:

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(i) Sketch graphs of the functions $H(x-1)$ and $H(x) - H(x-1)$;

(ii) The function $f(x)$ is defined in terms of the functions $a(x)$ and $b(x)$ as

$$f(x) = \begin{cases} a(x), & x \geq 0, \\ b(x), & x < 0. \end{cases}$$

Invent a single expression for $f(x)$ in terms of $a(x)$, $b(x)$ and $H(x)$.

5. We showed in lectures that any function defined over a suitable domain can be written as the sum of an even and an odd function. Decompose the following functions in this way, simplifying your answers where possible:

$$(a)\ \frac{1}{x+1}; \qquad (b)\ \left(\frac{1-x}{1+x}\right)^{1/2}; \qquad (c)\ \sin(x+1).$$

6. The exponential function is defined by the infinite series

$$f(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which is convergent for all values of x. By multiplying together the two series expansions for $f(x)$ and $f(y)$ and collecting the first few terms of the same degree (e.g. x^3, x^2y, xy^2 and y^3 are all of degree 3), verify that

$$f(x)f(y) = f(x+y). \qquad (1)$$

[To attempt a full proof, look at Progress Test 1 from 2006, on the website.]

7. In lectures we showed the exponential function $y = f(x)$ defined in Q6 is strictly increasing, and so has an inverse function, $y = g(x)$. Only using the property (1) and general results about inverses, establish the results:

$$g(uv) = g(u) + g(v); \qquad g\left(\frac{1}{u}\right) = -g(u); \quad g\left(\frac{u}{v}\right) = g(u) - g(v).$$

8. Find all solutions x to the following two equations:

$$(a)\ 4\sin^2 x - 5\cos x - 5 = 0; \qquad (b)\ 2\sec^2 x - \tan x - 3 = 0.$$

9. Use the identities

$$\sin(x+y) = \sin x \cos y + \sin y \cos x,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

to derive expressions for $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin\alpha + \sin\beta$, $\cos\alpha + \cos\beta$. And remember them.

10. Given that $t = \tan\frac{x}{2}$, show that

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}.$$