M1M1 Handout 2: Limits, Infinite Sequences and Series

This sheet has a brief summary of some important properties. More details next term.

(1) An **infinite sequence** is an ordered list of numbers or **terms**, which we write as $\{u_n\}$ for $n = 0,\ 1,\ 2\ldots$ The numbers u_n may be complex or real.

(2) The sequence $\{u_n\}$ has a **limit** U, if, for all n sufficiently large, $|u_n - U|$ is arbitrarily small. In that case we say the sequence **converges**, and write

$$u_n \to U \qquad \text{as} \quad n \to \infty\ , \qquad \text{or} \quad \lim_{n\to\infty} u_n = U\ .$$

If the sequence does not tend to a limit, we say it **diverges**. For example, if $u_n = 1/n$, then $u_n \to 0$ as $n \to \infty$. The sequence $\{r^n\}$, where r is given, has the limit

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad |r| < 1 \\ \infty & \text{if} \quad |r| > 1 \end{cases}$$

[Note: "∞" is not really a limit – it's one way a sequence can diverge. We may use the shorthand "$= \infty$" to mean "is arbitrarily large in modulus," but it's sloppy notation.]
If $|r| = 1$, the sequence does not converge unless $r = 1$, when it has the limit 1.

(3) Let $f(x)$ be a function of a real variable x. If for **every** sequence $\{x_n\} \to a$ as $n \to \infty$, the sequence $f(x_n) \to L$, then we say the limit of $f(x)$ as $x \to a$ exists, and $\lim_{x\to a} f(x) = L$. If **in addition** $f(a) = L$, we say $f(x)$ is **continuous** at a.

(4) Limits behave well under addition and multiplication, so that if $a_n \to A$, and $b_n \to B$, then $(a_n + b_n) \to (A + B)$ and $(a_n b_n) \to AB$. Also, if $b_n = f(a_n)$, where the function f is continuous at A, then $B = f(A)$.

(5) Polynomials in n are dominated as $n \to \infty$ by their largest power of n. Thus

$$\lim_{n\to\infty} \left(\frac{n^p + a_1 n^{p-1} + \ldots + a_p}{n^q + b_1 n^{q-1} + \ldots + b_q} \right) = \begin{cases} 0 & \text{if} \quad p < q \\ 1 & \text{if} \quad p = q \\ \infty & \text{if} \quad p > q \end{cases}$$

Exponentials dominate polynomials. For example, $\lim_{n\to\infty} n^{100}(1.01)^{-n} = 0$.

(6) Any **finite** number of terms cannot determine whether a sequence converges.

(7) An **infinite series** is the sum of all the terms in an infinite sequence. We interpret this infinite sum by defining a sequence of "partial sums," $s_n = \sum_{r=0}^{n} u_r$. If the sequence $\{s_n\} \to S$ (finite) as $n \to \infty$, we say that the infinite series converges to S, and write

$$\sum_{n=0}^{\infty} u_n = S\ .$$

Otherwise, we say that the series is **divergent.** The study of infinite series is a difficult topic, full of traps, especially when the terms may be either positive or negative.

(8) For a series $\sum\limits_{n=0}^{\infty} u_n$ to converge, it is **necessary** that $\lim\limits_{n\to\infty} u_n = 0$, i.e. the terms must tend to zero as $n \to \infty$. Note this condition is **not sufficient** to show convergence.

(9) **Absolute convergence**: We say that the series $\sum_{n=0}^{\infty} u_n$ is **absolutely convergent** if $\sum_{n=0}^{\infty} |u_n|$ is convergent. If a series is absolutely convergent then it is also convergent, and it is legitimate to reorder terms and perform other "sensible" operations on it.

(10) **The comparison test.** Suppose that a_n and b_n are real and $a_n \geqslant b_n \geqslant 0$. Then
(i) if $\sum\limits_{n=0}^{\infty} a_n$ is **convergent** then $\sum\limits_{n=0}^{\infty} b_n$ also converges.
(ii) if $\sum_{n=0}^{\infty} b_n$ is **divergent**, then so is $\sum_{n=0}^{\infty} a_n$.

(11) The **geometric series** $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$. If $r = 1$ it increases without limit, while if $r = -1$ it oscillates between 0 and 1.

(12) The **harmonic series** $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges. More generally, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^s}$, where s is real, converges iff $s > 1$. The behaviour of this series when s is complex is related to the most famous unsolved problem in mathematics, the **Riemann hypothesis**.

(13) **The ratio test.** Essentially using the comparison test with a geometric series we can show that if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = l \ , \qquad \text{then the series} \quad \sum_{n=0}^{\infty} a_n \begin{cases} \text{converges if } l < 1 \\ \text{diverges if } l > 1 \\ \text{uncertain if } l = 1 \end{cases}$$

This test is very useful, but sometimes the ratio does not tend to a limit.

(14) **Power series**: If $\{a_n\}$ is a given sequence, we can define a function $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ , \qquad \text{provided the series converges.}$$

We use the ratio test, defining $u_n = a_n x^n$, to infer that if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \qquad \text{then} \qquad \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}\, x^{n+1}}{a_n x^n} \right| = \frac{|x|}{R}$$

so that the series converges if $|x| < R$, diverges if $|x| > R$, and may do either if $|x| = R$. This critical value of $|x|$ is called the **Radius of Convergence** of the **power series**. R exists even if the ratio test fails. Sensible manipulation of the series is valid for $|x| < R$.

(15) If $|x| < R$, we can differentiate the power series for $f(x)$ term by term m times, and then evaluate at $x = 0$ to find $f^{(m)}(0) = a_m m!$. This leads to the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \ .$$

(16) A change of variable leads to the **Taylor series** for $f(x+h)$:

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + \ldots \frac{f^{(n)}(x)}{n!} h^n + O(h^{n+1}) \ .$$

This series is very useful when seeking approximations when h is small.