1. (a) Plot on the same diagram between $x = 0$ and $x = 1$ the two functions

$$f_1(x) = \sin \pi x \qquad f_2(x) = 4x(1 - x)$$

indicating carefully which curve is which, and justifying the distinction.

(b) Plot on the same diagram for $x > 0$ the functions

$$f_3(x) = \tfrac{2}{\pi} \tan^{-1} x, \qquad f_4(x) = \tanh \frac{2x}{\pi},$$

indicating carefully which curve is which, and justifying the distinction.

(c) Plot on the same diagram for $x > 0$ the functions

$$f_5(x) = \log x, \qquad f_6(x) = \frac{x-1}{x},$$

indicating carefully which curve is which, and justifying the distinction.

(d) Evaluate the limit

$$\lim_{x \to 1} \left[\frac{f_1(x) - f_6(x)}{f_5(x) - f_2(x)} \right].$$

(e) Calculate

$$\int_0^5 \Big[f_3(x) - f_4(x) \Big] \, dx.$$

(f) Extending the definitions of the functions to complex x in a standard way, find the imaginary x-values (i) for which f_3 is infinite and (ii) those for which f_4 is infinite.

2. (a) For a given constant λ, the function $y(x)$ obeys the differential equation and boundary conditions

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Obtain an expression for the $(n+2)^{th}$ derivative, $y^{(n+2)}(x)$ in terms of lower derivatives. Hence derive a series expansion for $y(x)$ about $x = 0$ when $\lambda = 3$, giving terms up to and including x^6.

(b) Find the radius of convergence of the series in part (a).

(c) Show that for certain special values of λ the infinite series terminates as a polynomial.

(d) If $y_0(x)$ is the solution when $\lambda = 0$, and $y_2(x)$ the solution when $\lambda = 6$, then evaluate the integral

$$\int_{-1}^{1} y_0 y_2 \, dx.$$

3. (a) The differentiable function $f(x)$ has a root at $x = \alpha$ and $f(0) \neq 0$. For a given constant k, a sequence of approximations to α is sought by means of the scheme

$$x_0 = 0, \qquad x_{n+1} = x_n + kf(x_n) \quad \text{for } n = 0, 1, 2 \ldots$$

Use the Mean Value Theorem to show that

$$|x_{n+1} - \alpha| = K_n |x_n - \alpha|,$$

for a value of K_n which depends on k and a suitable value of the derivative of f.

(b) What extra condition on $f(x)$ makes it possible to choose k to guarantee that $x_n \to \alpha$ as $n \to \infty$?

If it is known that, for all x, $0 < f'(x) < M$, what range of values of k will give convergence?

(c) Show that there always exists a value of k such that $x_n \to \alpha$ as $n \to \infty$, even if we don't know what it is.

(d) Newton's method is similar to part (a), except that it uses a different value of k each iteration,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_0 = 0.$$

If $f(x)$ is twice differentiable, use Taylor's series with a remainder to show that

$$x_{n+1} - \alpha = O(x_n - \alpha)^2.$$

Discuss whether we can expect $x_n \to \alpha$ in this case.

4. (a) For $n \geqslant 2$ express the integral

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

in terms of I_{n-2}, and hence find I_n for odd or even integers $n \geqslant 0$ as a finite series.

(b) Determine the limiting function

$$F(x) = \lim_{n\to\infty} \tan^n x \qquad \text{for} \quad 0 \leqslant x \leqslant \tfrac{1}{4}\pi.$$

Infer the limit of I_n as $n \to \infty$.

(c) Deduce from the above that

$$\pi = 4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} - \ldots$$

and obtain a similar series whose sum is $\log 2$.

(d) Obtain the series for π and $\log 2$ in part (c) directly, by considering the series for $\log(1 + x)$ and $1/(1 + x^2)$, which you may quote.

Solutions [ALL UNSEEN, except where explicitly stated]

1. (a) Both curves pass through $(0,0)$, $(1,0)$ with a maximum at $(1/2,1)$. To distinguish the two, one could note that $f_2'(0) = 4 > \pi = f_1'(0)$, or

$$\int_0^1 f_1\,dx = \tfrac{2}{\pi} < \tfrac{2}{3} = \int_0^1 f_2(x)\,dx \qquad \textbf{[3]}$$

(b) Each curve is odd and asymptotes to ± 1 and $f_3'(0) = f_4'(0) = 2/\pi$. However f_4 approaches 1 exponentially, whereas f_3 does it algebraically. For example, for large x consider $f_3' \sim 1/x^2$ whereas $f_4' \sim \text{sech}^2(2x/\pi) \sim \exp(-4x/\pi)$. **[3]**

(c) Each is infinite at $x = 0$, passes through $(1,0)$ with gradient 1. But for large x, f_5 slowly increases without limit but $f_6 \to 1$. Alternatively, f_6 tends to $-\infty$ more rapidly as x decreases to zero. **[3]**

(d) Using de l'Hôpital's rule, as the numerator and denominator are both zero at $x = 1$, the required limit is

$$\lim_{x\to 1}\left[\frac{f_1(x)-f_6(x)}{f_5(x)-f_2(x)}\right] = \lim_{x\to 1}\left[\frac{f_1'(x)-f_6'(x)}{f_5'(x)-f_2'(x)}\right] = \frac{-\pi-1}{1+4} = -\tfrac{1}{5}(\pi+1) \qquad \textbf{[3]}.$$

(e) The integrals are regular, so consider them separately.

$$\int_0^5 \frac{2}{\pi}\tan^{-1}x\,dx = \left[\frac{2x}{\pi}\tan^{-1}x\right]_0^5 - \frac{2}{\pi}\int_0^5 \frac{x}{1+x^2}\,dx = \frac{10}{\pi}\tan^{-1}5 - \frac{1}{\pi}\log 26.$$

$$\int_0^5 \tanh\frac{2x}{\pi}\,dx = \tfrac{\pi}{2}\Big[\log\cosh(2x/\pi)\Big]_0^5 = \frac{\pi}{2}\log\cosh(10/\pi)$$

Thus

$$\int_0^5 (f_3 - f_4)\,dx = \frac{10}{\pi}\tan^{-1}5 - \frac{1}{\pi}\log 26 - \frac{\pi}{2}\log\cosh(10/\pi). \qquad \textbf{[4]}$$

(f) Now $\tanh(ix) = i\tan(x)$ and $\tan(ix) = i\tanh(x)$. It follows that $f_4(x)$ is infinite whenever $2x/\pi = i(\frac{1}{2}\pi + n\pi)$ or at $x = i\frac{1}{4}\pi^2(1+2n)$ **[2]**
Furthermore there are no values of x such that $\tanh x = \pm 1$, so that $\tan^{-1}(\pm i)$ is also singular, so that $f_3(\pm i)$ is formally infinite. (Or consider the derivative $1/(1+x^2)$.) **[2]**

Total : 20

2. (a) Differentiating n times by Leibniz, we have

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - 2n(n-1)/2y^{(n)} - 2xy^{(n+1)} - 2ny^{(n)} + \lambda y^{(n)} = 0,$$

or

$$(1-x^2)y^{(n+2)} - (2n+2)xy^{(n+1)} = [n(n+1)-\lambda]y^{(n)}. \qquad \textbf{[4]}$$

Substituting $x = 0$, we have

$$y^{(n+2)}(0) = [n(n+1)-\lambda]y^{(n)}(0). \qquad \textbf{[1]}$$

Now as $y'(0) = 0$, all odd derivatives vanish at 0, and the series only has even terms. By repeated use of the above result, we have when $\lambda = 3$

$$y(0) = 1, \quad y''(0) = -3, \quad y^{(4)}(0) = 3y''(0) = -9, \quad y^{(6)}(0) = 17y^{(4)}(0) = -9 * 17$$

so that

$$y(x) + \sum_{n=0}^{\infty} \frac{y^{(n)}(0)x^n}{n!} = 1 - \frac{3}{2}x^2 - \frac{3}{8}x^4 - \frac{17}{80}x^6 + O(x^8) \qquad \textbf{[5]}$$

(b) Looking at the ratio of adjacent terms, we have

$$\left|\frac{y^{(n+2)}(0)x^{n+2}/(n+2)!}{y^{(n)}(0)x^n/n!}\right| = \frac{|n(n+1)-\lambda|x^2}{(n+1)(n+2)} \to x^2 \qquad \text{as } n \to \infty.$$

By the ratio test, the series converges for $|x| < 1$ so the radius of convergence is 1. **[3]**

(c) Now if $\lambda = k(k+1)$ for some positive integer k. then $y^{(k+2)}(0) = 0$ as are all higher derivatives. It follows that the series terminates as a polynomial (of order k – not required). **[3]**

(d) When $\lambda = 0$, the series terminates after the first term, so that $y_0(x) = 1$. When $\lambda = 6$, this corresponds to $k = 2$. The solution is $y_2(x) = 1 - 3x^2$. So the reuired integral is

$$\int_{-1}^{1} 1(1-3x^2)\,dx = \Big[x - x^3\Big]_{-1}^{1} = 0. \qquad \textbf{[4]}$$

Total : **20**

3. (a) We have $f(\alpha) = 0$. The MVT states that there exists a value ξ_n between x_n and α such that

$$f'(\xi_n)(x_n - \alpha) = f(x_n) - f(\alpha) = f(x_n).$$

It follows that

$$x_{n+1} - \alpha = x_n - \alpha + kf'(\xi_n)[(x_n - \alpha)] = [1 + kf'(\xi_n)](x_n - \alpha).$$

so we may define

$$K_n = |1 + kf'(\xi_n)|, \qquad \Longrightarrow \qquad |x_{n+1} - \alpha| = K_n|x_n - \alpha|.$$ **[5]**

(b) Clearly, $K_n < 1$ iff $-2 < kf'(\xi_n) < 0$. However, $f'(\xi_n)$ may vary in sign for different n, in which case, no single value of k suffices. If we require that f' is of single sign over the domain of interest, then we can choose k to be of opposite sign. We must then choose $|k|$ small enough such that $1 + kf' > -1$. Thus if $0 < f' < M$, we will choose $0 > k > -2/M$. **[5]**

(c) If we choose $k = \alpha/f(x_0)$, then $x_1 = \alpha$ and then $x_2 = \alpha$ and so on. Clearly then $x_n \to \alpha$. As this value of k depends on α, we don't know what it is, however. **[4]**

(d) The Taylor series with remainder states for some η and μ

$$f(x_n) = f(\alpha) + (x_n - \alpha)f'(\alpha) + \tfrac{1}{2}(x_n - \alpha)^2 f''(\eta) \qquad \text{and} \qquad f'(x_n) = f'(\alpha) + (x_n - \alpha)f''(\mu)$$

Substituting in, we have

$$x_{n+1} = x_n - \frac{(x_n - \alpha)f'(\alpha) + O(x_n - \alpha)^2}{f'(\alpha) + O(x_n - \alpha)} = \alpha + O(x_n - \alpha)^2$$

Or as required

$$(x_{n+1} - \alpha) = O(x_n - \alpha)^2 < A(x_n - \alpha)^2,$$ **[4]**

for some A. Thus provided $|x_n - \alpha|$ is small enough, $|x_{n+1} - \alpha|$ will be smaller still. So we expect Newton's method to converge provided our starting point ($x = 0$) is sufficiently close to the actual root ($x = \alpha$). **[2]**

Total : 20

4. (a) Writing $\tan^2 x = \sec^2 x - 1$, we have

$$I_n = \int_0^{\pi/4} \sec^2 x \tan^{n-2} x\, dx - I_{n-2} = \frac{1}{n-1}\Big[\tan^{n-1} x\Big]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \quad \textbf{[3]}$$

Thus if n is even

$$I_n = \frac{1}{n-1} - \frac{1}{n-3} + \ldots - (-1)^{n/2}\left(1 - \int_0^{\pi/4} 1\, dx\right)$$

Or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots \pm \frac{1}{n-1} \mp I_n \quad \textbf{[2]}$$

If n is odd, then

$$I_n = \frac{1}{n-1} - \frac{1}{n-3} + \ldots - (-1)^{(n-1)/2}\left(\frac{1}{2} - \int_0^{\pi/4} \tan x\, dx\right)$$

Or

$$\Big[-\log\cos x\Big]_0^{\pi/4} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \ldots \pm \frac{1}{n-1} \mp I_n,$$

and so

$$\tfrac{1}{2}\log 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \ldots \pm \frac{1}{n-1} \mp I_n \quad \textbf{[3]}$$

(b) Now $0 \leqslant \tan x \leqslant 1$ over this range. Furthermore, as $n \to \infty$, $r^n \to 0$ for $0 < r < 1$. It follows that

$$F(x) = 0 \qquad \text{for } x \neq \tfrac{1}{4}\pi, \quad F(\tfrac{1}{4}\pi) = 1. \quad \textbf{[2]}$$

We infer that $I_n \to 0$ as $n \to \infty$. **[1]**

(c) Rearranging the above series, we have therefore in the limit as $n \to \infty$,

$$\pi = 4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} \ldots = \sum_{m=1}^{\infty} \frac{4(-1)^{m-1}}{2m-1} \quad \textbf{[2]}$$

and

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \textbf{[3]}$$

(d) [**SEEN**] We have $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$ and substituting $x = 1$ gives the correct formula. Now

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \ldots \qquad \Longrightarrow \qquad \int_0^1 \frac{dx}{1+x^2} = \Big[x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots\Big]_0^1$$

giving

$$\tan^{-1} 1 = \tfrac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \quad \textbf{[4]}$$

Total : 20