Imperial College London

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

M1M1

Mathematical Methods 1

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The function $y(x)$ for $x > 0$ satisfies the ODE and boundary conditions

$$xy'' = y, \qquad y(0) = 0, \qquad y'(0) = 1.$$

(a) Assuming the n^{th} derivative $y^{(n)}(x)$ exists for every n, find a relation between $y^{(n+1)}(0)$ and $y^{(n)}(0)$.

(b) Hence find the Maclaurin series for $y(x)$, giving the general term.

(c) Calculate the Radius of Convergence of this series.

(d) Writing $t = 2\sqrt{x}$ and $y = tu(t)$, obtain a differential equation for $u(t)$.

(e) Determine whether the functions $y(x)$ and $u(t)$ are even, odd or neither.

(f) Allowing x to be complex, we define $\sqrt{x}$ to have a positive real part if possible. As x starts from $x = 4$ and moves anticlockwise round the circle $|x| = 4$, what path does t (as defined in part (d)) describe in the complex t-plane?

(g) Obtain a relation between $y(1)$ and $y'(1)$ and $\int_0^1 y(x)\,dx$.

2. (a) The functions f and g are continuous and $g > 0$ on the closed interval $[a, b]$. Prove that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx, \tag{1}$$

for some (unknown) ξ in $[a, b]$. [You may use various properties of integrals and continuous functions without proof, but state clearly what properties you are assuming.]

(b) Assuming that (1) applies, deduce that

$$\tfrac{1}{4}\pi < \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}} < \tfrac{1}{2}\pi. \tag{2}$$

(c) Why might part (a) not apply in this case? Justify, briefly, why the result (2) is nevertheless true.

(d) Express $\sin^2\theta$ and $\cos^2\theta$ in terms of $t = \tan\theta$.
Hence or otherwise, evaluate the integral in part (b) exactly, and verify that (2) does indeed hold.

3. (a) A triangle, T, has vertices $(0,\,0)$, $(1,\,1)$ and $(1,\,2)$. By using first vertical strips, and then horizontal strips, calculate in two ways the double integral

$$\iint_T 2xy\,dA.$$

(b) If ∂T denotes the boundary of T, find by any method the value of the integral

$$\int_{\partial T} ds$$

where s is the usual arclength.

(c) Obtain a general relation between the 2nd derivatives d^2y/dx^2 and d^2x/dy^2 and the gradient dy/dx. Hence establish a relation between the curvatures

$$K_1 = \frac{d^2y/dx^2}{(1+(dy/dx)^2)^{3/2}} \qquad \text{and} \quad K_2 = \frac{d^2x/dy^2}{(1+(dx/dy)^2)^{3/2}}.$$

For the case $y = \cosh x$ calculate K_1 and K_2 and verify the connection between them.

4. (a) A real cubic polynomial is written

$$f(x) = x^3 + ax^2 + bx + c,$$

where a, b and c are real constants. Find a constant λ such that

$$g(y) \equiv f(y+\lambda) = y^3 + dy + e,$$

and define the constants d and e.

By considering the location of turning points of $g(y)$ or otherwise, determine when $f(x)$ has complex roots. (You may leave conditions in terms of d and e).

(b) Find the derivative of $f(x)$ in part (a) **from first principles**, that is, from the formal definition of the derivative.

(c) Suppose now that a, b and c may be **complex**, but f nevertheless has 3 real roots r_1, r_2 and r_3. Find the conditions on a, b and c for this to occur.

[Hint: First express a, b and c in terms of r_i using the Fundamental Theorem of Algebra or otherwise.]

Solutions [ALL UNSEEN, except where explicitly stated]

1. (a) Differentiating n times by Leibniz, we have

$$xy^{(n+2)} + ny^{(n+1)} = y^{(n)}.$$

Thus at $x = 0$, using that all derivatives exist (and so are finite)

$$ny^{(n+1)}(0) = y^{(n)}(0). \qquad \mathbf{[2]}$$

(b) This equation holds when $n = 0$. When $n = 1, 2, 3\ldots$ we have $y''(0) = 1$, $y'''(0) = 1/2$, $y''''(0) = 1/3!$ and by inspection, $y^{(n)}(0) = 1/(n-1)!$. Thus

$$y = \sum_{n=0}^{\infty} \frac{x^n y^{(n)}(0)}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}. \qquad \mathbf{[3]}$$

(c) The ratio of adjacent terms is

$$\frac{x^{n+1}/(n!(n+1)!)}{x^n/((n-1)!n!)} = \frac{x}{(n+1)n} \to 0 \qquad \text{as } n \to \infty \text{ for all } x.$$

As this limit is always less than 1, we conclude the series converges for all x, i.e. has infinite Radius of Convergence. **[2]**

(d) If $y(x) = tu(t)$, we have $y' = (tu' + u)dt/dx = (tu' + u)/x^{1/2}$. Differentiating again,

$$y'' = \frac{(tu'' + 2u')}{x} - (tu' + u)/2x^{3/2} \qquad \Longrightarrow \qquad xy'' = tu'' + 2u' - (tu' + u)/t.$$

Thus

$$xy'' = y \qquad \Longrightarrow \qquad t^2u'' + tu' - (t^2 + 1)u = 0. \qquad \mathbf{[3]}$$

(e) From part (b) we see $y(x)$ is neither even nor odd. As $x = t^2/4$, $u(t)$ consists only of odd powers of t and so is clearly odd. **[1+1]**

(f) If x is complex, we have to define $\sqrt{x}$ carefully. If $x = re^{i\theta}$ with $-\pi < \theta \leqslant \pi$, then $\sqrt{x} = r^{1/2}e^{i\theta/2}$, ensures that the real part is positive or zero. If $x = 4e^{i\theta}$, then $t = 4e^{i\theta/2}$. This describes only half of the circle of radius 4 centre $t = 0$. As x traverses the circle anticlockwise, t starts from 4 and traverses a quarter of the same circle anticlockwise until $\theta = \pi$, when it jumps from $+4i$ to $-4i$, and then continuing anticlockwise until $t = 4$. **[4]**

(g) From the original ODE,

$$\int_0^1 y\,dx = \int_0^1 xy''\,dx = \Big[xy'\Big]_0^1 - \int_0^1 y'\,dx = y'(1) - y(1). \qquad \mathbf{[4]}$$

integrating by parts and using the boundary conditions.

Total : 20

2. (a) [**SEEN with** $g \equiv 1$ **only**] A continuous function on a closed interval is bounded and attains its bounds, so that there exist max and min values of f, M and m such that

$$M \geqslant f(x) \geqslant m \qquad \Longrightarrow \qquad Mg(x) \geqslant f(x)g(x) \geqslant mg(x),$$

since $g \geqslant 0$. Inequalities can be integrated, so that

$$M \int_a^b g(x)\,dx \geqslant \int_a^b f(x)g(x)\,dx \geqslant m \int_a^b g(x)\,dx.$$

Thus

$$M \geqslant \phi \geqslant m, \qquad \text{where} \quad \phi = \int_a^b fg\,dx \Big/ \int g(x)\,dx\ .$$

Now a continuous function attains all values between its maximum and minimum values somewhere. Thus there is at least one value of x where $f(x) = \phi$. Call this value $x = \xi$, and we deduce

$$\int_a^b f(x)g(x)\,dx = f(\xi) \int_a^b g(x)\,dx, \qquad [\mathbf{7}]$$

as required. [Note $\int g\,dx \neq 0$].

(b) Setting $f = 1/(1+x^2)$ and $g(x) = 1/\sqrt{1-x^2}$, we have $\int_0^1 g(x)\,dx = \sin^{-1} 1 = \frac{1}{2}\pi$. Now $f(x)$ is a decreasing function, so it attains its maximum, $M = 1$, when $x = 0$ and its minimum, $m = \frac{1}{2}$ when $x = 1$. It follows that the required integral lies between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$ as required. **[3]**

(c) However, the conditions of part (a) required both f and g to be continuous on the closed interval, whereas g is discontinuous (singular) at $x = 1$. So the theorem as stated does not strictly apply. However, it does apply for $b = 1 - \varepsilon$ for any $0 < \varepsilon \ll 1$. Furthermore, the integrand has a square-root singularity only, $fg \simeq \frac{1}{2\sqrt{2}}(1-x)^{-1/2}$ near $x = 1$. This is integrable, and so we expect the limit as $\varepsilon \to 0$ to exist. **[3]**

(d) If $t = \tan\theta$, then $1 + t^2 = \sec^2\theta$ and so

$$\cos^2\theta = \frac{1}{1+t^2} \qquad \text{and} \qquad \sin^2\theta = \frac{t^2}{1+t^2}. \qquad [\mathbf{1\ mark}]$$

Substituting $x = \sin\theta$, we have

$$I = \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\cos\theta(1+\sin^2\theta)} = \int_0^{\pi/2} \frac{d\theta}{1+\sin^2\theta}.$$

Using the hint in the question, we substitute $t = \tan\theta$, to obtain

$$I = \int_0^\infty \frac{dt/(1+t^2)}{1+t^2/(1+t^2)} = \int_0^\infty \frac{dt}{1+2t^2} = \frac{1}{\sqrt{2}}\Big[\tan^{-1}(t\sqrt{2})\Big]_0^\infty = \frac{\pi}{2\sqrt{2}}.$$

This lies between $\pi/2$ and $\pi/4$ as expected. **[6]**

Total : 20

3. (a) The boundaries are the lines $y = x$, $y = 2x$ and $x = 1$. Using vertical strips, we fix x and first integrate between $y = x$ and $y = 2x$, and then between $x = 0$ and $x = 1$:

$$I = \int_0^1 \left(\int_x^{2x} 2xy \, dy \right) dx = \int_0^1 \left[xy^2 \right]_x^{2x} dx = \int_0^1 3x^3 \, dx = \tfrac{3}{4}. \qquad [\mathbf{3}]$$

If we fix y, we see from the shape of the triangle that we have different upper boundaries depending whether $y < 1$ or $y > 1$. The lower boundary is $x = \frac{1}{2}y$, while the upper is either $x = 1$ or $x = y$

$$I = \int_0^1 \left(\int_{y/2}^{y} 2xy \, dx \right) dy + \int_1^2 \left(\int_{y/2}^{1} 2xy \, dx \right) dy. \qquad [\mathbf{3}]$$

This time,

$$I = \int_0^1 y(y^2 - y^2/4) dy + \int_1^2 y(1 - y^2/4) dy = \tfrac{3}{16} + \tfrac{3}{2} - \tfrac{15}{16} = \tfrac{12}{16} = \tfrac{3}{4}. \qquad [\mathbf{2}]$$

These agree, as expected.

(b) The integral is just the total perimeter, which is clearly $\sqrt{2} + 1 + \sqrt{5}$. [**2**]

(c) [**Relation between y'' and x'' SEEN on problem sheet.**]

We have

$$\frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1}.$$

On differentiation of this expression with respect to y and use of the chain rule

$$\frac{d^2x}{dy^2} = \frac{dx}{dy} \frac{d}{dx} \left(\frac{dy}{dx} \right)^{-1} = -\frac{dx}{dy} \left(\frac{dy}{dx} \right)^{-2} \frac{d^2y}{dx^2} = -\left(\frac{dy}{dx} \right)^{-3} \frac{d^2y}{dx^2}. \qquad [\mathbf{3}]$$

Thus if we substitute in K_2, for example,

$$K_2 = \frac{-d^2y/dx^2 (dy/dx)^{-3}}{(1 + (dx/dy)^2)^{3/2}} = -\frac{d^2y/dx^2}{[(1 + (dy/dx)^2)]^{3/2}} = -K_1. \qquad [\mathbf{2}]$$

If $y = \cosh x$, then

$$K_1 = \frac{\cosh x}{(1 + \sinh^2 x)^{3/2}} = \frac{1}{\cosh^2 x}. \qquad [\mathbf{2}]$$

Now

$$\frac{dx}{dy} = \frac{1}{\sqrt{y^2 - 1}}, \qquad \frac{d^2x}{dy^2} = \frac{-y}{(y^2 - 1)^{3/2}}$$

So $1 + (dx/dy)^2 = y^2/(y^2 - 1)$. Thus

$$K_2 = \frac{-y/(y^2 - 1)^{3/2}}{y^3/(y^2 - 1)^{3/2}} = -\frac{1}{y^2} = -K_1. \qquad [\mathbf{3}]$$

These agree, as expected.

Total : **20**

4. (a) Now

$$f(y+\lambda) = (y+\lambda)^3 + a(y+\lambda)^2 + b(y+\lambda) + c = y^3 + y^2(3\lambda + a) + y(3\lambda^2 + 2a\lambda + b) + f(\lambda).$$

So we choose $\lambda = -a/3$ and then

$$d = b - \tfrac{1}{3}a^2, \qquad e = \tfrac{2}{27}a^3 - \tfrac{1}{3}ab + c. \qquad \textbf{[3]}$$

Now $g(y) = y^3 + dy + e$ has complex roots iff $f(x)$ does. If conversely $g(y)$ has three real roots, then between any two such roots by Rolle's theorem there must be a zero of $g'(y) = 3y^2 + d$. This has roots iff $d \leqslant 0$, namely $y = \pm\alpha$ where $\alpha = \sqrt{-d/3}$. So a sufficient condition for complex roots to exist is $d > 0$ or $a^2 < 3b$.

If there are turning points, by considering the general shape of the graph, we will have 3 real roots if g takes opposite signs at the turning points. If this happens, the product of the two values is negative. Now $g(\pm\alpha) = \pm 2\alpha^3 + e$. So for real roots, we require

$$g(\alpha)g(-\alpha) \leqslant 0 \qquad \Longrightarrow \quad 27e^2 \leqslant -4d^3.$$

Combining these results, we have complex roots if $d > 0$ or $27e^2 > -4d^3$ or equivalently just $27e^2 > -4d^3$. **[9]**

(b) By definition

$$f'(x) = \lim_{\varepsilon\to 0}\left[\frac{f(x+\varepsilon) - f(x)}{\varepsilon}\right] = \lim_{\varepsilon\to 0}\left[\frac{3\varepsilon x^2 + a2\varepsilon x + b\varepsilon + O(\varepsilon^2)}{\varepsilon}\right] = 3x^2 + 2ax + b.$$

[3]

(c) If $f(x)$ has roots r_i, then

$$f(x) = x^3 + ax^2 + bx + c = (x - r_1)(x - r_2)(x - r_3).$$

This means $c = -r_1r_2r_3$, $a = -(r_1 + r_2 + r_3)$ and $b = r_1r_2 + r_2r_3 + r_3r_1$. It follows that a b and c must be real for all the roots to be real. Hence using part (a), we require $d \leqslant 0$ and $-4d^3 \geqslant 27e^2$ or equivalently $-4d^3 \geqslant 27e^2$. **[5]**

Total : 20