Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2013

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Use the Mean Value Theorem to show that $1 > \log 2 > \frac{1}{2}$.

(b) Evaluate the integral

$$\int_0^{\pi} \frac{dx}{1+\sin x}.$$

(c) Find the inverse of the function

$$f(x) = \frac{x}{x^2-1} \qquad \text{for} \quad |x| < 1.$$

(d) Find $y(x)$ satisfying the ODE

$$y' \cos^2 x + y = 1$$

together with the boundary condition $y(0) = 0$.

(e) Evaluate the limit

$$\lim_{x\to\infty} \left[e^{-x} \log(x) + (x^2 + \tanh x)\log(1+x^{-3}) + \cos\left(\frac{1+\pi x}{\pi + x}\right)\right].$$

2. (a) By considering the effect of very small increases in the polar coordinates (r, θ) to $(r+\delta r, \theta + \delta\theta)$, obtain expressions for the area element dA and the arc-length element ds in terms of dr and $d\theta$. Deduce that the curve defined by $r = f(\theta)$ for $0 < \theta < 2\pi$ has an area

$$A = \int_0^{2\pi} \tfrac{1}{2} f^2 \, d\theta,$$

and give a corresponding formula for L, the total length of the curve.

(b) Sketch the curve defined by $r = 1 + \sin 2\theta$ for $0 < \theta < 2\pi$. [Do not try to do this in Cartesian coordinates.] Identify the points furthest from the origin and the intersections with the x- and y-axes.

(c) Calculate the area, A, enclosed by the curve.
Show also, that the curve length L can be written

$$L = \int_0^{2\pi} \left(5 + 2s - 3s^2\right)^{1/2} d\theta \qquad \text{where} \quad s = \sin 2\theta.$$

Do not attempt to evaluate the integral, but show that $L < 8\pi/\sqrt{3}$.

(d) Find dy/dx for the curve in part (b) as a function of θ and show that it takes the value 2 when the curve crosses the positive y-axis.

3. (a) The function $y(x)$ is given by

$$y(x) = \log\left[\sqrt{x^2+1} - x\right].$$

Find the derivative y', and show that

$$(1+x^2)y'' = -xy'.$$

Deduce that

$$n^2 y^{(n)}(0) + y^{(n+2)}(0) = 0,$$

where $y^{(n)}$ denotes the n'th derivative.

Hence find the Taylor series for $y(x)$ about $x = 0$, giving terms up to x^8.

(b) Calculate the radius of convergence of the series in part (a). Explain, without proof, why this value is to be expected.

(c) Does your series suggest that $y(x)$ is even, odd or neither? Prove that result, directly from the definition of $y(x)$.

4. (a) The function $C(\theta)$ is defined by the infinite series

$$C(\theta) = 1 - \tfrac{1}{2}\cos\theta + \tfrac{1}{4}\cos 2\theta - \ldots = \sum_{n=0}^{\infty}\left(-\tfrac{1}{2}\right)^n \cos(n\theta)$$

By writing C as the real part of a complex function, or otherwise, show that

$$C = \frac{1+\frac{1}{2}\cos\theta}{\frac{5}{4}+\cos\theta} \equiv \frac{1+\frac{1}{2}\mu}{\frac{5}{4}+\mu}, \qquad \text{writing} \quad \mu = \cos\theta.$$

(b) Find all (complex) values of θ such that $\cos\theta = -\frac{5}{4}$.
Hence infer what the radius of convergence would be if C were expanded as a power series in θ. [N.B. Experiment on a computer suggests it is between 3 and 4.]

(c) Writing $\mu = \cos\theta$, find all complex values of μ for which $|C| = 1$, and sketch them in the complex μ-plane.

Solutions [ALL UNSEEN, except where explicitly stated]

1. (a) Consider $f(x) = \log x$ in $[1,2]$. Then $f' = 1/x$, and by the MVT, there exists a ξ in $1 < \xi < 2$ (or $\frac{1}{2} < \xi^{-1} < 1$) such that

$$\frac{\log 2 - \log 1}{2-1} = \frac{1}{\xi} \qquad \Longrightarrow \qquad \tfrac{1}{2} < \log 2 < 1. \qquad \textbf{[3]}$$

(Alternatively, use $[1/2, 1]$. Proofs not using the MVT will not earn full credit.)

(b) Substituting $t = \tan \frac{1}{2}x$, we have

$$\int_0^\pi \frac{dx}{1+\sin x} = \int_0^\infty \frac{2dt/(1+t^2)}{1+2t/(1+t^2)} = \int_0^\infty \frac{2dt}{(1+t)^2} = \left[\frac{-2}{1+t}\right]_0^\infty = 2. \qquad \textbf{[4]}$$

(c) Solving $y = f(x)$ for x, we have $yx^2 - x - y = 0$ or $x = (1 \pm \sqrt{1+4y^2})/(2y)$. **[2]**
For $|x| < 1$ we must take the $-$ sign (e.g. consider what happens for small y). Thus the inverse function is

$$f^{-1}(y) = \frac{1-\sqrt{1+4y^2}}{2y}. \qquad \textbf{[2]}$$

(d) The integrating factor for this linear 1st order ODE is $\exp(\int \sec^2 x) = \exp(\tan x)$. **[1]**
Thus

$$\left[e^{\tan x} y\right]' = \sec^2 x \, e^{\tan x}$$

or

$$ye^{\tan x} = e^{\tan x} + A \qquad \Longrightarrow \qquad y = 1 + Ae^{-\tan x}, \qquad \textbf{[3]}$$

where A is a constant. Imposing $y(0) = 0$ gives $A = -1$, and

$$y = 1 - e^{-\tan x}. \qquad \textbf{[2]}$$

(e) As $x \to \infty$, $e^x \gg \log(x)$ and so the first term tends to zero. We have $\tanh x \to 1$ and $\log(1+x^{-3}) \sim x^{-3}$, so that the second term is approximately $(x^2+1)/x^3 \to 0$. The third term tends to $\cos(\pi) = -1$. So the limit is -1. **[3]**

Total : 20

2. (a) [**SEEN**] A small change in θ corresponds to a length $r\delta\theta$, which is perpendicular to the radial line. Thus the area corresponding to small changes in both r and θ is roughly rectangular with area $r\delta r\delta\theta$. We therefore expect that the area inside the closed curve $r = f(\theta)$ will be the sum of all these infinitesimal rectangles or

$$A = \int_0^{2\pi}\int_0^{f(\theta)} r dr\, d\theta = \int_0^{2\pi} \tfrac{1}{2} f^2\, d\theta. \qquad [\mathbf{2}]$$

Similarly the arc length is approximately the hypotenuse of a right-angled triangle, with $\delta s^2 = r^2\delta\theta^2 + \delta r^2$. Thus the total length L, is

$$L = \int_0^{2\pi} \frac{ds}{d\theta} d\theta = \int_0^{2\pi} \left[f^2 + \left(\frac{df}{d\theta}\right)^2 \right]^{1/2} d\theta. \qquad [\mathbf{2}]$$

(b) See attached sketch. r is maximum when $f' \equiv 2\cos 2\theta = 0$. Clearly the maximum is when $\sin 2\theta = +1$ and the minimum when $\sin 2\theta = -1$. **[for sketch, 5]**

(c) The area is

$$A = \tfrac{1}{2}\int_0^{2\pi} (1+\sin 2\theta)^2 d\theta = \tfrac{3}{2}\pi. \qquad [\mathbf{1}]$$

Now

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1+\sin 2\theta)^2 + (2\cos 2\theta)^2 = 1 + 2\sin 2\theta + \sin^2 2\theta + 4(1-\sin^2 2\theta)$$

So

$$L = \int_0^{2\pi} \left[5 + 2\sin 2\theta - 3\sin^2 2\theta\right]^{1/2} d\theta \qquad [\mathbf{2}]$$

Now $5 + 2s - 3s^2 = 5 - 3(s-1/3)^2 + 1/3 \leqslant 16/3$. Thus

$$L < \int_0^{2\pi} (16/3)^{1/2} d\theta = 8\pi/\sqrt{3}. \qquad [\mathbf{3}]$$

(d) We have $x = r\cos\theta = \cos\theta(1+\sin 2\theta)$ Thus

$$\frac{dx}{d\theta} = -\sin(\theta)(1+\sin 2\theta) + 2\cos\theta\cos 2\theta$$

Similarly

$$\frac{dy}{d\theta} = \cos(\theta)(1+\sin 2\theta) + 2\sin\theta\cos 2\theta$$

Thus

$$\frac{dy}{dx} = \frac{\cos(\theta)(1+\sin 2\theta) + 2\sin\theta\cos 2\theta}{-\sin(\theta)(1+\sin 2\theta) + 2\cos\theta\cos 2\theta} \qquad [\mathbf{4}]$$

or equivalent forms. The curve crosses the positive y-axis when $\theta = \frac{1}{2}\pi$ giving

$$\frac{dy}{dx} = \frac{-2}{-1} = 2. \qquad [\mathbf{1}]$$

Total : 20

3. (a) We have

$$y' = \left(\frac{1}{\sqrt{x^2+1}-x}\right)\left(\frac{x}{\sqrt{x^2+1}}-1\right) = \frac{-1}{\sqrt{x^2+1}} \qquad [\mathbf{1}]$$

Thus

$$y'' = \frac{+x}{(x^2+1)^{3/2}} \qquad \Longrightarrow \qquad (x^2+1)y'' = -xy' \qquad [\mathbf{2}]$$

as required. Using Leibniz' rule, we have

$$(x^2+1)y^{(n+2)} + n2xy^{(n+1)} + \tfrac{1}{2}n(n-1)2y^{(n)} = -xy^{(n+1)} - ny^{(n)}.$$

Evaluating this at $x=0$, we have

$$y^{(n+2)}(0) = y^{(n)}(0)\left[-n-n(n-1)\right] = -n^2 y^{(n)}(0), \qquad [\mathbf{4}]$$

as required.

Now $y(0) = \log(1) = 0$, and $y'(0) = -1$ as above. It follows that $y^{(2k)}(0) = 0$ for integers k, by repeated use of the above equation. Similarly, we have

$$y^{(3)}(0) = (-1)(-1) = 1, \qquad y^{(5)}(0) = -3^2, \qquad y^{(7)}(0) = (3^2)(5^2).$$

Now

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!}x^n = -x + \frac{x^3}{3!} - \frac{3^2}{5!}x^5 + \frac{3^2 5^2}{7!}x^7 + \ldots = -x + \tfrac{1}{6}x^3 - \tfrac{3}{40}x^5 + \tfrac{5}{112}x^7 + O(x^9).$$

[**4**]

(b) Since the even terms in the series are zero, the ratio of adjacent terms in the series can be written as

$$\left|\frac{y^{(n+2)}(0)/(n+2)!\; x^{n+2}}{y^{(n)}(0)/n!\; x^n}\right| = \left|\frac{-n^2x^2}{(n+2)(n+1)}\right| \to |x^2| \quad \text{as} \quad n \to \infty.$$

By the ratio test, the series converges if $|x| < 1$ and diverges if $|x| > 1$. The radius of convergence is 1. [**4**]

This is to be expected because the derivatives become infinite as $x \to \pm i$ in the complex x-plane. So the circle of convergence cannot have radius larger than unity. [**2**]

(c) The series only has odd terms, so it seems that $y(x)$ is odd. From the definition, we have

$$y(x) + y(-x) = \log\left[\sqrt{x^2+1}-x\right] + \log\left[\sqrt{x^2+1}+x\right]$$

$$= \log\left[\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)\right] = \log(x^2+1-x^2) = 0$$

It follows that $y(-x) = -y(x)$ and $y(x)$ is odd. [**3**]

Total : 20

4. (a) Writing $z = e^{i\theta}$, we have $\cos n\theta = \Re e[z^n]$, so that

$$C = \Re e\left[1 - \tfrac{1}{2}z + \tfrac{1}{4}z^2 + \ldots\right] = \Re e\left[\frac{1}{1+\frac{1}{2}z}\right] = \Re e\left[\frac{1+\frac{1}{2}e^{-i\theta}}{(1+\frac{1}{4}+\frac{1}{2}e^{i\theta}+\frac{1}{2}e^{-i\theta})}\right].$$

The denominator is now real, so we deduce that

$$C = \frac{1+\frac{1}{2}\cos\theta}{\frac{5}{4}+\cos\theta}, \qquad \text{as required.}$$ **[6]**

[Another method: consider $(\cos\theta)C$ and use $\cos A\cos B$ formula.]

(b) If $\cos\theta = -5/4$ then

$$-\tfrac{5}{2} = e^{i\theta} + e^{-i\theta} \qquad \Longrightarrow \qquad e^{i\theta} = -\tfrac{5}{4} \pm \tfrac{1}{2}\sqrt{\tfrac{25}{4} - 4} = -2 \text{ or } -\tfrac{1}{2}.$$

Thus as $\log(-1) = (2n+1)i\pi$ for integers n, we have

$$i\theta = \pm\log(-2) = \pm[\log 2 + i(2n+1)\pi].$$ **[6]**

Or $\theta = \pm i\log 2 \pm (2n+1)\pi$. At each of these values of θ, C becomes infinite. Those nearest to the origin are $\theta = \pm i\log 2 \pm \pi$. Each of these 4 values is a distance $R = \sqrt{(\log 2)^2 + \pi^2}$ from the origin, so we expect this to be the Radius of Convergence of the series for $C(\theta)$. **[3]**

(c) If $|C| = 1$, then $|1 + \frac{1}{2}\mu| = |\frac{5}{4} + \mu|$ and letting $\mu = x + iy$, we have

$$(1+\tfrac{1}{2}x)^2 + \tfrac{1}{4}y^2 = (\tfrac{5}{4}+x)^2 + y^2 \qquad \Longrightarrow \qquad \tfrac{3}{4}x^2 + \tfrac{3}{2}x + \tfrac{3}{4}y^2 + \tfrac{9}{16} = 0.$$

Rearranging, we have

$$(x+1)^2 + y^2 = \tfrac{1}{4}, \qquad \text{a circle of radius } \tfrac{1}{2} \text{ with centre } \mu = -1\ .$$ **[5]**

Total : 20