Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. For a given real constant b, the function $f(x)$ takes the form

$$f(x) = \frac{x^2 + 2bx + 1}{x^2 - 1} \qquad \text{for} \quad x^2 \neq 1.$$

(a) If $y = f(x)$ and y is real, show that x is real only if some inequality relating y and b holds. From this inequality, show that $f(x)$ takes all real values as x varies, provided $b^2 \geqslant 1$.

(b) Express $f(x)$ in partial fractions.

(c) In general, what is the meaning of the limit (if it exists)

$$L(d) = \lim_{x \to d} \left[\frac{g(x) - g(d)}{x - d} \right] \ ?$$

Evaluate this limit if $g(x) \equiv f(x)$ and $d^2 \neq 1$.
For which values of b does $L(d) = 0$ for some real value(s) of d?

(d) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, give the value of a_n for each value of n.

(e) Give rough sketches of the curve $y = f(x)$, in the 3 cases $b > 1$, $b = 1$, $0 < b < 1$. Verify that your sketches are consistent with parts (a) and (c).
[NB Part (e) of the question is worth more than the others.]

(f) For which values of b does the integral

$$\int_0^{100} f(x)\,dx \qquad \text{exist?}$$

2. (a) Find all complex numbers z such that $|z+i| = 2|z-i|$.

(b) Express the function $\tan x$ in terms of $u \equiv e^{ix}$.

Hence, if p is real and $p > 1$, find all complex numbers z which satisfy

$$\tan z = ip.$$

(c) While preparing this exam, I produced the following incorrect argument which falsely "proves" that there are no solutions to part (b):

(1) For all A and B, we have

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(2) For all real y, we have $\tan(iy) = i \tanh y$.

(3) Suppose $z = x + iy$, where x and y are real, and that $\tan z = ip$, with $p > 1$. Then

$$\tan(x+iy) = \frac{\tan x + i \tanh y}{1 - i \tan x \tanh y} = ip.$$

(4) Multiplying up, we have

$$\tan x + i \tanh y = ip + p \tan x \tanh y.$$

Equating imaginary parts, we have $\tanh y = p$

(5) But for real y, $|\tanh y| < 1$, so there are no solutions to this problem if $p > 1$.

What was wrong with my argument? (The actual solution from part (b) may offer a hint.)

3. (a) For non-negative integers m and n, we denotes by $F(m,n)$ the integral

$$F(m,n) = \int_0^1 x^m (1-x)^n \, dx.$$

Show that for $m \geqslant 1$,

$$F(m,n) = \left(\frac{m}{m+n+1}\right) F(m-1,n)$$

and hence derive an expression for $F(m,n)$ in terms of factorials.

Deduce that

$$\int_0^{\pi/2} \sin^{11}\theta \cos^{13}\theta \, d\theta = \frac{(5!)(6!)}{2(12!)}.$$

(b) For a given function $f(x)$, the function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} - y\tan x = f(x)$$

with the boundary condition $y(1) = 0$.

Obtain the solution in terms of a definite integral.

If $f(x) = x^3(1-x)^6 \sec x$, show that

$$y(0) = -\frac{1}{840}.$$

4. (a) Evaluate the limit

$$\lim_{x\to 0}\left[\frac{\int_0^x \sin(t^2)\, dt}{\sin x + \cos x - \log(1+x) - \exp(x^3)}\right].$$

(b) If $t = \tanh x$, show that

$$x = \tfrac{1}{2}\log\left[\frac{1+t}{1-t}\right].$$

Use the Mean Value Theorem for the function $\log(1+x)$ to show that for $t > 0$

$$t > \log(1+t) > \frac{t}{1+t}.$$

Show also that for $0 < t < 1$,

$$t < -\log(1-t) < \frac{t}{1-t}$$

and deduce that for $0 < t < 1$

$$\frac{t - \frac{1}{2}t^2}{1-t} > \tanh^{-1} t > \frac{t + \frac{1}{2}t^2}{1+t}.$$

Solutions [ALL UNSEEN, except where explicitly stated]

1. (a) If $y = f(x)$ then

$$y(x^2-1) = x^2 + 2bx + 1 \qquad \Longrightarrow \qquad x^2(1-y) + 2bx + (1+y) = 0.$$

If this has a real root for x, we must have

$$(2b)^2 \geqslant 4(1-y)(1+y) \iff b^2 \geqslant 1 - y^2.$$

If this constraint is to be satisfied for all y, we must have $|b| \geqslant 1$. **[2 marks]**

(b)

$$f(x) = 1 + \frac{2bx+2}{x^2-1} = 1 + \frac{b+1}{x-1} + \frac{b-1}{x+1}.$$ **[1 mark]**

[Thus as $x \to \pm\infty$, $f \sim 1 + 2b/x$ which is greater/less than 1 according to the sign of b/x. Near $x = 1$, $f \to \pm\infty$, as determined by the sign of $\frac{b+1}{x-1}$ and so on. This is useful for the curve plotting.]

(c) In general, $L = g'(d)$, the derivative of g at $x = d$. If $g = f$, this is **[1 mark]**

$$L = f' = -\frac{(b+1)}{(d-1)^2} - \frac{(b-1)}{(d+1)^2}.$$

Now for $L = 0$ we must have

$$\left(\frac{d+1}{d-1}\right)^2 = \frac{1-b}{1+b}.$$

This is positive provided $|b| < 1$. (From part (b), $|b| = 1$ does not work. Do not insist they check this.) So $L(d) = 0$ for some d if $|b| < 1$. **[2 marks]**

(d) Expanding the denominator, we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = -(1 + 2bx + x^2)(1 + x^2 + x^4 + \ldots)$$

By inspection, multiplying the two brackets gives $a_0 = -1$ while if n is odd, $a_n = -2b$. If n is even but non-zero, $a_n = -1 - 1 = -2$. **[3 marks]**

(e) See attached sketches. The general shape suffices. i.e. approach asymptotes from the right side, maximum and minimum somewhere, all values attained between $x = \pm 1$ if $|b| > 1$. As $b \to -b$, $x \to -x$, so sketches for $b < 0$ are mirror images of $b > 0$ (not required). Particularly good sketches (e.g. giving Max and Min points explicitly) may compensate for a small error in another sketch. **[9 marks]**

(f) From part (b) the integrand has singularities like $(x-d)^{-1}$ at $d = \pm 1$ but is otherwise continuous. $x = -1$ is outside the range, but there is a term which is non-integrable at $x = 1$. So the integral does not exist unless this term disappears, i.e. if $b = -1$. So integral exists iff $b = -1$. **[2 marks]**

Total : 20

2. (a) Writing $z = x + iy$, so that $z \pm i = x + i(y \pm 1)$ we have

$$|z+i| = 2|z-i| \qquad \Longrightarrow \qquad x^2 + (y+1)^2 = 4(x^2 + (y-1)^2)$$

$$3x^2 + 3y^2 - 10y + 3 = 0 \qquad \Longrightarrow \qquad x^2 + (y - \tfrac{5}{3})^2 = \tfrac{16}{9}.$$ **[5 marks]**

This a circle centre $(0, \frac{5}{3})$ radius $\frac{4}{3}$.

(b) We have $2\cos x = e^{ix} + e^{-ix} = u + u^{-1}$ and $2i \sin x = e^{ix} - e^{-ix}$ Thus

$$\tan x = -i \left(\frac{u^2 - 1}{u^2 + 1} \right).$$

Thus if $\tan z = ip$ and $v = e^{iz}$.

$$\frac{v^2 - 1}{v^2 + 1} = -p \qquad \Longrightarrow \qquad v^2 = \frac{1-p}{1+p} = -\frac{p-1}{p+1} e^{2n\pi i}.$$ **[2 marks]**

where n is any integer. Thus if $p > 1$

$$v \equiv e^{iz} = i\sqrt{\frac{p-1}{p+1}} e^{n\pi i}.$$ **[3 marks]**

Taking logarithms, we have

$$iz = \log i + \tfrac{1}{2} \log \left(\frac{p-1}{p+1} \right) + n\pi i$$

so that as $i = \exp(\frac{1}{2}\pi i)$,

$$z = (n + \tfrac{1}{2})\pi - \tfrac{1}{2} i \log \left(\frac{p-1}{p+1} \right).$$ **[5 marks]**

(c) From the above solution we see that $x = (n + \frac{1}{2})\pi$. It follows that $\tan x$ is infinite and we had better be careful.

In step (3) we multiplied up by the denominator, which we see is actually infinite. Multiplying by infinity is like dividing by zero – it must not be done. Instead, if we take the limit as $\tan x \to \infty$ in step 3, we have $i/\tanh y = ip$, so that $y = \tanh^{-1}(1/p)$. This gives $y = \frac{1}{2} \log \frac{p+1}{p-1}$ which agrees with part (a). [This last part is not required – an observation that $\tan x$ is infinite and we cannot multiply by infinity suffices.] **[5 marks]**

Total : 20

3. (a)

$$\int_0^1 x^m(1-x)^n\,dx = \left[-\frac{x^m(1-x)^{n+1}}{n+1}\right]_0^1 + \frac{m}{n+1}\int_0^1 x^{m-1}(1-x)^{n+1}\,dx$$
$$= \frac{m}{n+1}\int_0^1 (x^{m-1}-x^m)(1-x)^n\,dx.$$

Thus

$$F(m,n) = \frac{m}{n+1}(F(m-1,n)-F(m,n)) \qquad \Longrightarrow \qquad F(m,n) = \frac{m}{m+n+1}F(m-1,n).$$

[4 marks]

Continuing down to $F(0,n)$,

$$F(m,n) = \left(\frac{m}{m+n+1}\right)\left(\frac{m-1}{m+n}\right)\cdots\left(\frac{1}{n+2}\right)\int_0^1 (1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$$

[4 marks]

Putting $x=\sin^2\theta$, we have $dx = 2\sin\theta\cos\theta\,d\theta$ and

$$F(m,n) = 2\int_0^{\pi/2} \sin^{2m+1}\theta\cos^{2n+1}\theta\,d\theta.$$

Thus

$$\int_0^{\pi/2}\sin^{11}\theta\cos^{13}\theta\,d\theta = \tfrac{1}{2}F(5,6) = \frac{(5!)(6!)}{2(12)!}.$$

[2 marks]

(b) The integrating factor is $I=\exp[-\int\tan x\,dx] = \exp[\log(\cos x)] = \cos x$. **[2 marks]**

$$\frac{d}{dx}(y\cos x) = \cos x f(x).$$

So that if we impose $y(1)=0$

$$y(x) = \frac{1}{\cos x}\int_1^x \cos(t)f(t)\,dt.$$

[4 marks]

If $f(x) = x^3(1-x)^6\sec x$, then

$$y(0) = \frac{1}{1}\int_1^0 t^3(1-t)^6\,dt = -F(3,6) = -\frac{3!}{10(9)(8)(7)} = -\frac{1}{840}.$$

[4 marks]

Total : 20

4. (a) $\sin t^2 = t^2 + O(t^4)$ and

$$\sin x+\cos x-\log(1+x)-\exp(x^3) = x-\tfrac{1}{6}x^3+1-\tfrac{1}{2}x^2-(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3)-(1+x^3)+O(x^4)$$

Thus

$$\lim_{x\to 0}\left[\frac{\int_0^x \sin(t^2)\,dt}{\sin x+\cos x-\log(1+x)-\exp(x^3)}\right] = \lim_{x\to 0}\left[\frac{\frac{1}{3}x^3+O(x^5)}{x^3\left[-\frac{1}{6}-\frac{1}{3}-1\right]+O(x^4)}\right] = -\tfrac{2}{9}.$$

(Or could use de l'Hôpital's rule.) **[5 marks]**

(b) If $t = \tanh x$, then

$$t = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$$

This implies

$$e^{2x} = \frac{1+t}{1-t} \qquad \Longrightarrow \qquad x = \tfrac{1}{2}\log\frac{1+t}{1-t}.$$ **[SEEN, 2 marks]**

The MVT for $\log(1+x)$ between $x=0$ and $x=t$, states that there exists a ξ, for $0<\xi<t$ such that

$$\frac{\log(1+t)-\log(1)}{t-0} = \frac{1}{1+\xi}.$$ **[2 marks]**

But as $1/(1+x)$ is a decreasing function,

$$1 > \frac{1}{1+\xi} > \frac{1}{1+t} \qquad \Longrightarrow \qquad 1 > \frac{\log(1+t)}{t} > \frac{1}{1+t}.$$

As $t>0$ we can multiply the inequality by it, to obtain

$$t > \log(1+t) > \frac{t}{1+t}.$$ **[3 marks]**

Using the same function over the interval $(-t,\,0)$ [Alternatively, use $\log(1-t)$], we have there is an η in $-t<\eta<0$ such that

$$\frac{\log(1)-\log(1-t)}{0-(-t)} = \frac{1}{1+\eta} \qquad \text{and} \qquad \frac{1}{1-t} > \frac{1}{1+\eta} > 1.$$

Thus for $1>t>0$

$$\frac{t}{1-t} > -\log(1-t) > t.$$ **[5 marks]**

Adding the two inequalities, in the range where both apply $(0<t<1)$ and noting that $2\tanh^{-1}(t) = \log(1+t)-\log(1-t)$,

$$t+\frac{t}{1-t} > 2\tanh^{-1} t > t + \frac{t}{1+t}$$

or

$$\frac{t-\frac{1}{2}t^2}{1-t} > \tanh^{-1} t > \frac{t+\frac{1}{2}t^2}{1+t}.$$ **[3 marks]**

[Note: some may get into trouble subtracting inequalities, $A>B$ and $C>D$ does not mean $A-C>B-D$ but rather $A-D>B-C$. As all results are given, care must be taken when marking that proper arguments are used.]

Total : 20