Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2011

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The functions $f(x)$ and $g(x)$ are defined for all real x by

$$f(x) = \sin|x| - |\sin x|, \qquad g(x) = \int_0^x f(t)\,dt.$$

(a) Are $f(x)$ and $g(x)$ even, odd or neither?

(b) Obtain simplified expressions for $f(x)$ in the ranges $2n\pi < x < (2n+1)\pi$ and $(2n+1)\pi < x < (2n+2)\pi$, where n is a positive integer..

(c) Evaluate $f(-4\pi/3)$ and $g(-4\pi/3)$.

(d) For which values of x is the derivative of $f(x)$ defined?

(e) Evaluate, if possible, the limit

$$\lim_{x \to \pi} \left[\frac{f(x)}{x - \pi} \right].$$

(f) Sketch the functions $f(x)$ and $g(x)$ over the interval $(-4\pi, 4\pi)$.

(g) What is the range of $g(x)$?

(h) Is $g(x)$ invertible? If so, find its inverse.

(i) Calculate $g(2n\pi)$, where n is a positive integer and determine whether or not the integral

$$\int_4^\infty \frac{1}{g(x)}\,dx$$

exists. (Do not attempt to evaluate the integral exactly.)

2. For an infinitely differentiable function $f(x)$, the error in the $(n+1)$-term Taylor expansion about $x = a$ is E_n, where

$$f(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + E_n,$$

where $f^{(n)}$ denotes the n-th derivative. Use induction to show that

$$E_n = \int_a^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\,dt.$$

If $f^{(n+1)}(t)$ varies continuously over the interval $[a, x]$ attaining its maximum values M and minimum m somewhere in this range, prove that

$$E_n = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some $\xi \in [a, x]$.

Write down the first three terms in the Taylor series about $x = 1$ for the function $f(x) = e^{-x}$. Determine the smallest value of n which ensures that the error $|E_n|$ is less than 10^{-3} over the entire interval $(1, 2)$.

3. (a) If $y = \exp[\sin^{-1} x]$, show that

$$(1 - x^2)y'' - xy' - y = 0.$$

Differentiating this equation n times, deduce that

$$y^{(n+2)}(0) = (n^2 + 1)y^{(n)}(0),$$

where $y^{(n)}$ denotes the n'th derivative of y.

Hence write down the first 5 terms in the Maclaurin series for y.

Infer the radius of convergence of the infinite series. (It may help to consider it as the sum of an odd series and an even series.)

(b) The complex-valued function $y(x)$ obeys the differential equation

$$xy' = y(ix - 1) - 1.$$

Find the general solution for $y(x)$.

Find also the particular solution which is finite everywhere, and use de l'Hôpital's rule to evaluate its limiting value as $x \to 0$.

4. (a) Simplify $(z^n + z^{-n})$ if $z = e^{i\theta}$ for some real θ.
By considering $(z + z^{-1})^8$, obtain an expression for $\cos^8\theta$ in the form

$$\cos^8\theta = \sum_{n=0}^{8} a_n \cos n\theta,$$

where the coefficients a_n are to be found. Hence evaluate

$$\int_{\pi/2}^{\pi} \cos^8\theta \, d\theta,$$

in terms of the coefficients a_n.

(b) Defining I_n for positive integers n by

$$I_n = \int_{\pi/2}^{\pi} \cos^n x \, dx,$$

obtain an expression relating I_n to I_{n-2}. Hence calculate I_8, and compare your answer with part (a).

(c) Using the techniques of either part (a) or (b), derive a formula for

$$\frac{2^{2k}(k!)^2}{(2k)!} I_{2k},$$

where k is a positive integer.

Solutions

1. **[ALL UNSEEN]**

(a)

$$f(-x) = \sin|-x| - |\sin(-x)| = \sin|x| - |-\sin x| = \sin|x| - |\sin x| = f(x)$$

Thus $f(x)$ is even (and so for much of this question we can restrict attention to $x \geqslant 0$). Also, substituting $s = -t$,

$$g(-x) = \int_0^{-x} f(t)\,dt = \int_0^{x} f(-s)(-ds) = -\int_0^{x} f(s)\,ds = -g(x).$$

Thus $g(x)$ is odd. **[2 marks]**

(b) If $x \in (2n\pi,\ (2n+1)\pi)$ then $x > 0$, $\sin x > 0$ and $f(x) = 0$. Whereas, if $x \in ((2n+1)\pi,\ (2n+2)\pi)$, then $\sin x < 0$ and $f(x) = 2\sin x$. **[2 marks]**

(c) Using the above $f(-4\pi/3) = \sin(4\pi/3) - |\sin(4\pi/3)| = -2\sin(\pi/3) = -\sqrt{3}$. Also,

$$g(-\tfrac{4}{3}\pi) = -g(\tfrac{4}{3}\pi) = -2\int_{\pi}^{4\pi/3} \sin t\,dt = -2(\cos\pi - \cos(4\pi/3)) = -2(-1+1/2) = 1.$$

[2 marks]

(d) Using our knowledge of the modulus and sine functions, f' is defined except possibly at points where either x or $\sin x$ changes sign, i.e. at $x = n\pi$ for integers n. Now at $x = 0$ the function is flat so this is not a problem. But near $x = n\pi$ for $n > 0$, $f(x) = \sin x - |\sin x|$, which switches between 0 and $2\sin x$ near $x = n\pi$. Thus $f'(x)$ switches from 0 to $2\cos x$ and this last is ± 2. We conclude, the derivative is discontinuous at these points, and hence $f(x)$ is not differentiable. So $f'(x)$ exists except for $x = \pm n\pi$, where $n = 1, 2, \ldots$ **[2 marks]**

(e) Since $f(\pi) = 0$, the limit in question is the derivative of $f(x)$ at $x = \pi$. Hence by the previous part, this limit does not exist. Alternatively, the limit for $x > \pi$ differs from that for $x < \pi$. **[2 marks]**

(f) See sketches. $f(x)$ alternates between 0 and $2\sin x$, and $f \leqslant 0$. Thus $g(x)$ is constant in places and $A - 2\cos x$ in places for suitable constants A. **[4 marks]**

(g) $g(x)$ is continuous, and $g(x) \to -\infty$ as $x \to \infty$ and $g(x) \to +\infty$ as $x \to -\infty$. Thus the range of $g(x)$ is $(-\infty,\ \infty)$. **[1 mark]**

(h) As $g(x)$ is constant for some ranges, it is not invertible - for example, any x in $|x| < \pi$ has $g(x) = 0$. **[2 marks]**

(i) For $x \geqslant 4$, $g(x) \neq 0$, so the integrand is finite. Now

$$g(2n\pi) = n\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right] = \Big[-2n\cos x\Big]_{\pi}^{2\pi} = -4n$$

Furthermore, from the graph, $1/g \leqslant 1/g((2n-2)\pi)$ for $(2n-2)\pi < x < 2n\pi$. Therefore,

$$\int_4^{\infty} \frac{dx}{g(x)} = \int_4^{2\pi} + \int_{2\pi}^{4\pi} + \ldots \leqslant \int_4^{2\pi} \frac{dx}{g(x)} - \tfrac{1}{4}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots\right)$$

Therefore the integral diverges to $-\infty$. **[3 marks]**
(Details not required – argued observation that $1/g \sim 1/x$ would suffice).

Total : 20

2. **[First part is seen – see the attached annotated handout.]** **[12 marks]**
Now the n'th derivative of e^{-x} is $(-1)^n e^{-x}$, and so

$$e^{-x} = e^{-1} - e^{-1}(x-1) + \tfrac{1}{2}e^{-1}(x-1)^2 + \ldots \qquad \textbf{[3 marks]}$$

Now over the interval $(1,2)$, $x-1<1$ and $e^{-x} < e^{-1} < 1/2$. Thus $|E_n| < e^{-1}/n!$. We want $|E_n| < 10^{-3}$, which is the case for $n=5$, as $e.6! = 720e > 1000$. **[5 marks]**

Total : 20

3. **[ALL UNSEEN. They have not seen a complex ODE in this course.]**

(a) We have

$$y' = \exp(\sin^{-1} x)(1-x^2)^{-1/2} \qquad \Longrightarrow \qquad y = y'(1-x^2)^{1/2}.$$

Thus

$$y' = (1-x^2)^{1/2}y'' - x(1-x^2)^{-1/2}y' \qquad \Longrightarrow \qquad (1-x^2)y'' - xy' - y = 0. \;\; \textbf{[3 marks]}$$

Using Leibnitz' rule,

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - n(n-1)y^{(n)} - xy^{(n+1)} - ny^{(n)} - y^{(n)} = 0$$

Or

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2+1)y^{(n)} = 0.$$

Putting $x=0$, we have $y^{(n+2)}(0) = (n^2+1)y^{(n)}(0)$. **[3 marks]**
Now $y(0)=1$, $y'(0)=1$ and so $y''(0)=1$, $y'''(0)=2$, $y^{(4)}(0)=5$. Hence

$$y(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{5}{24}x^4 + \ldots \qquad \textbf{[3 marks]}$$

If we write $y(x)$ as the sum of an even series and an odd series, the ratio of adjacent terms in each series is

$$\left|\frac{y^{(n+2)}(0)x^{n+2}/(n+2)!}{y^{(n)}(0)x^n/n!}\right| = x^2\frac{n^2+1}{(n+2)(n+1)}.$$

As $n \to \infty$, this has the limit x^2 and so each series has radius of convergence 1. It follows that the full series has R. of C 1 also. **[2 marks]**

(b) Equation is linear. The Integrating Factor is

$$I = \exp\left[\int\left(\frac{1}{x} - i\right)dx\right] = \exp(\log x - ix) = xe^{-ix}. \qquad \textbf{[2 marks]}$$

Thus the ODE is

$$\left[xye^{-ix}\right]' = -e^{-ix} \qquad \Longrightarrow \qquad xye^{-ix} = -ie^{-ix} + C.$$

So the general solution is

$$y = \frac{Ce^{ix} - i}{x}. \qquad \textbf{[3 marks]}$$

This is infinite at $x = 0$ unless the numerator vanishes, which requires $C = i$. So the particular solution is

$$y = \frac{i}{x}(e^{ix} - 1). \qquad \textbf{[2 marks]}$$

As $x \to 0$, by de l'Hôpital's rule, $y \to (i/1)(i) = -1$. **[2 marks]**

Total : 20

4. **[Part a unseen, part b seen recurrence relation for integral between $(0, \pi/2)$.]**

(a) $z^n + z^{-n} = e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$. **[1 mark]**
Now by a binomial expansion,

$$(z + z^{-1})^8 = z^8 + 8z^6 + 28z^4 + 56z^2 + 70 + 56z^{-2} + 28z^{-4} + 8z^{-6} + z^{-8}$$

Thus

$$(2\cos\theta)^8 = 2\left[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\right]$$

or

$$\cos^8\theta = \tfrac{1}{128}\cos 8\theta + \tfrac{1}{16}\cos 6\theta + \tfrac{7}{32}\cos 4\theta + \tfrac{7}{16}\cos 2\theta + \tfrac{35}{128}. \qquad \textbf{[5 marks]}$$

Integrating between $\frac{1}{2}\pi$ and π, the only non zero term is due to the constant a_0, so that

$$\int_{\pi/2}^{\pi}\cos^8\theta\,d\theta = a_0\tfrac{1}{2}\pi = \tfrac{35}{128}\tfrac{1}{2}\pi \qquad \textbf{[3 marks]}$$

(b) Integrating by parts,

$$I_n \equiv \int_{\pi/2}^{\pi} \cos^n x \, dx = \int_{\pi/2}^{\pi} \cos x \cos^{n-1} x \, dx$$

$$= \Big[\sin x \cos^{n-1} x \Big]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x (n-1)(-\sin x) \cos^{n-2} x \, dx$$

$$= (n-1) \int_{\pi/2}^{\pi} (1 - \cos^2 x)(\cos^{n-2} x) \, dx = (n-1) \left[I_{n-2} - I_n \right].$$

Rearranging, we have

$$I_n = \left(\frac{n-1}{n} \right) I_{n-2} \qquad \textbf{[5 marks]}$$

Thus in particular,

$$I_8 = \tfrac{7}{8} I_6 = \tfrac{7}{8}\tfrac{5}{6}\tfrac{3}{4}\tfrac{1}{2} I_0 = \tfrac{35}{128}\tfrac{1}{2}\pi, \qquad \textbf{[2 marks]}$$

which agrees with part (a).

(c) Using part (a), the coefficient a_0 in the binomial expansion is ${}_{2k}C_k = (2k)!/(k!)^2$. Thus

$$2^{2k} I_{2k} = \int_{\pi/2}^{\pi} (z + z^{-1})^{2k} \, d\theta = a_0 \tfrac{1}{2}\pi = \frac{(2k)!}{(k!)^2} \tfrac{1}{2}\pi$$

Thus

$$\frac{2^{2k}(k!)^2}{(2k)!} I_{2k} = \tfrac{1}{2}\pi. \qquad \textbf{[4 marks]}$$

Alternatively, using part (b)

$$I_{2k} = \left(\frac{2k-1}{2k} \right) \left(\frac{2k-3}{2k-2} \right) \dots \left(\frac{3}{4} \right) \left(\frac{1}{2} \right) \tfrac{1}{2}\pi = \frac{(2k)!}{(2k)^2 (2k-2)^2 (2k-4)^2 \dots 2^2} \tfrac{1}{2}\pi$$

$$= \frac{(2k)!}{2^{2k} k^2 (k-1)^2 (k-2)^2 \dots} \tfrac{1}{2}\pi = \frac{(2k)!}{2^{2k}(k!)^2} \tfrac{1}{2}\pi$$

once more, giving the same result.

Total : 20