M1M1 Summer 2010: Solutions

The paper is entirely unseen.

1 (a) If $y = \coth x = (e^{2x}+1)/(e^{2x}-1)$, then $e^{2x} = (y+1)/(y-1)$. Thus

$$x = \coth^{-1}(y) = \tfrac{1}{2}\log\left[\frac{y+1}{y-1}\right] \qquad \text{so} \qquad \coth^{-1}(x) = \tfrac{1}{2}\log\left[\frac{x+1}{x-1}\right]. \qquad \textbf{[3 marks]}$$

(b) See sketch **[3 marks]**

(c) Differentiating $y' = \sinh x/\sinh x - \cosh^2 x/\sinh^2 x = 1 - y^2$. Differentiating once more, $y'' = -2yy' = -2y(1-y^2) = 2y^3 - 2y$. **[2 marks]**

(d) We have

$$\coth x = \coth a + \coth'(a)(x-a) + \tfrac{1}{2}\coth''(a)(x-a)^2 + \tfrac{1}{6}\coth'''(a)(x-a)^3 + \ldots$$

We know $\coth a = 2$, and by part (c) $\coth'(a) = 1 - 4 = -3$, $\coth''(a) = 2(8-2) = 12$. Hence

$$\coth x = 2 - 3(x-a) + 6(x-a)^2 + O(x-a)^3. \qquad \textbf{[3 marks]}$$

(e) We know $\cosh(i\theta) = \cos(\theta)$ and $\sinh(i\theta) = i\sin\theta$. So $\cosh(i\theta) = 0 \iff \cos\theta = 0 \iff \theta = (n+\frac{1}{2})\pi$ for integers n. (And note $\sin\theta \neq 0$ when $\cos\theta = 0$). So $\coth x = 0$ when $x = (n+\frac{1}{2})\pi i$. **[2 marks]**

(f) As $x \to \infty$,

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = (1+e^{-2x})(1-e^{-2x})^{-1} = 1 + 2e^{-2x} + O(e^{-4x}).$$

Thus $\log\sqrt{\coth x} \simeq \frac{1}{2}\log(1+2e^{-2x}) \simeq e^{-2x}$. Furthermore, writing $t = 1/x$, as $t \to 0$,

$$\coth t = \frac{2 + O(t^2)}{1 + t - (1-t) + O(t^2)} = \frac{1}{t} + O(t).$$

So

$$\lim_{x\to\infty}\left[\frac{\log\left(\log\sqrt{\coth x}\right)}{\coth(1/x)}\right] = \lim_{x\to\infty}\left[\frac{\log(e^{-2x})}{1/(1/x)}\right] = -2. \qquad \textbf{[4 marks]}$$

(g) The integrand is continuous except possibly where $\coth x$ or $\coth(\coth x)$ is infinite, i.e. at $x = 0$ and as $x \to \infty$. Near $x = 0$, $\coth x \simeq 1/x$ from (f) and $\coth(\coth x) \simeq 1$. So there is no singularity at $x = 0$. As $x \to \infty$, we know from (f) that $\coth x \simeq 1 + 2e^{-2x}$, so that $\coth(\coth x) - \coth 1 \to 0$ exponentially fast. So there is no problem at infinity. As the integrand is continuous and tends to zero faster than $1/x$ as $x \to \infty$, we conclude that the integral exists. [Allow the flawed argument that the integrand goes to zero at infinity, without worrying about the rate.] **[3 marks]**

[Total 20]

2. By inspection of the first few values of n (could prove by induction, but not required)

$$\frac{d^n}{dx^n}\left(\frac{1}{x+c}\right) = \frac{(-1)^n n!}{(x+c)^{n+1}}. \qquad \textbf{[2 marks]}$$

$x^2 - 2x\cos\alpha + 1 = (x-\cos\alpha)^2 + \sin^2\alpha$ so the roots are $x = \cos\alpha \pm i\sin\alpha$ Thus

$$\begin{aligned}\frac{2\sin\alpha}{x^2-2x\cos\alpha+1} &= \frac{2\sin\alpha}{(x-\cos\alpha-i\sin\alpha)(x-\cos\alpha+i\sin\alpha)} \\ &= \frac{-i}{x-\cos\alpha-i\sin\alpha} + \frac{i}{(x-\cos\alpha+i\sin\alpha)}\end{aligned} \qquad \textbf{[3 marks]}$$

Therefore

$$f^{(n)}(x) = (-1)^n n!\left[\frac{-i}{(x-\cos\alpha-i\sin\alpha)^{n+1}} + \frac{i}{(x-\cos\alpha+i\sin\alpha)^{n+1}}\right].$$

So

$$\begin{aligned}\frac{f^{(n)}(0)}{n!} &= (-1)^n\left[-i(-e^{i\alpha})^{-(n+1)} + i(-e^{-i\alpha})^{-(n+1)}\right] = (-1)^2 i\left[e^{-(n+1)i\alpha} - e^{(n+1)i\alpha}\right] \\ &= i(-2i\sin[(n+1)\alpha]) = 2\sin[(n+1)\alpha.\end{aligned}$$

[5 marks]

So from the Maclaurin series,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} 2\sin[(n+1)\alpha]x^n \qquad \text{as required.} \qquad \textbf{[2 marks]}$$

Substituting $x = \cos\alpha$, we have $f(\cos\alpha) = 2\sin\alpha/(1-\cos^2\alpha) = 2/\sin\alpha$. Thus

$$\frac{1}{\sin\alpha} = \sum_{n=0}^{\infty}(\cos\alpha)^n\sin[(n+1)\alpha]. \qquad \textbf{[3 marks]}$$

Substituting $\alpha = \frac{1}{3}\pi$, so $\cos\alpha = 1/2$, $\sin\alpha = \sqrt{3}/2$, while $\sin[(n+1)\pi/3]$ cycles between $\sqrt{3}/2$, $\sqrt{3}/2$, 0, $-\sqrt{3}/2$, $-\sqrt{3}/2$ and 0 for $n = 0, 1, 2, 3, 4, 5$ and then repeats. Thus

$$\begin{aligned}\sum_{n=0}^{\infty}(\cos\tfrac{1}{3}\pi)^n\sin[(n+1)\tfrac{1}{3}\pi] &= \frac{\sqrt{3}}{2}\left[1-(\tfrac{1}{2})^3+(\tfrac{1}{2})^6+\ldots\right] + \frac{\sqrt{3}}{2}\left[\tfrac{1}{2}-(\tfrac{1}{2})^4+(\tfrac{1}{2})^7+\ldots\right] \\ &= \frac{\sqrt{3}}{2}\left[\frac{1}{1-(-\frac{1}{8})} + \frac{1/2}{1-(-\frac{1}{8})}\right] = \frac{\sqrt{3}}{2}\frac{3}{2}\frac{8}{9} = \frac{2}{\sqrt{3}} = \frac{1}{\sin\frac{1}{3}\pi}\end{aligned} \qquad \textbf{[5 marks]}$$

[Total 20]

3. (a) The MVT states that if $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists a value ξ such that $a < \xi < b$ and

$$f(b) - f(a) = (b - a)f'(\xi). \qquad \textbf{[2 marks]}$$

If $f'(x) = 0$ for all x in (a, b), let c and d be any two values such that $a \leqslant c < d \leqslant b$. Then $f'(x) = 0$ for all x in (c, d). And the MVT therefore implies that $f(d) - f(c) = 0$, i.e. $f(d) = f(c)$. Thus $f(x)$ is constant for all x in $[a, b]$. **[3 marks]**

(b) Differentiating, we obtain a separable ODE:

$$f' = f^2 + 1 \qquad \Longrightarrow \qquad \int \frac{df}{f^2 + 1} = \int dx \qquad \Longrightarrow \qquad \tan^{-1} f = x + C \qquad \textbf{[5 marks]}$$

where C is a constant. Therefore $f(x) = \tan(x + C)$. However, the original equation requires $f(a) = 0 + a = a$. Thus $\tan(a + C) = a$ or $C = -a + \tan^{-1}(a)$. We conclude

$$f(x) = \tan(x - a + \tan^{-1}(a)) = \frac{a + \tan(x - a)}{1 - a\tan(x - a)}. \qquad \textbf{[4 marks]}$$

Alternatively, substituting in the original equation we have

$$\tan(x + C) = \int_a^x \left[\sec^2(t + C) - 1\right] dt + x \qquad \Longrightarrow \qquad 0 = -\tan(a + C) + a \qquad \text{etc.}$$

(For forgetting C award a total of **4 marks.**)

(c) Equation is linear, with integrating factor

$$I = \exp\left[\int \tan x \, dx\right] = \exp[-\log\cos x] = \sec x. \qquad \textbf{[2 marks]}$$

Thus equation becomes

$$\frac{d}{dx}(y \sec x) = \sec^2 x \qquad \Longrightarrow \qquad y \sec x = \tan x + C \qquad \Longrightarrow \qquad y = \sin x + C\cos x.$$

[4 marks]
[Total 20]

4. (a) Integrating by parts,

$$\int_0^\infty u^4 e^{-u}\,du = \Big[-u^4 e^{-u}\Big]_0^\infty + 4\int_0^\infty u^3 e^{-u}\,du = 4\int_0^\infty u^3 e^{-u}\,du$$

as we know $u^n e^{-u} \to 0$ as $u \to \infty$. Continuing similarly,

$$\int_0^\infty u^4 e^{-u}\,du = (4)(3)(2)\int_0^\infty e^{-u}\,du = 24.$$ [**3 marks**]

(b) We note $d/dx[-\log(x)]^b = -b/x[-\log(x)]^{b-1}$. Integrating by parts, we have

$$\int_0^1 x^a\,[-\log x]^b\,dx = \left[\frac{x^{a+1}}{a+1}[-\log x]^b\right]_0^1 - \int_0^1 \frac{x^{a+1}}{a+1}\left(\frac{-b}{x}\right)[-\log x]^{b-1}\,dx.$$

The first term vanishes provided $b > 0$. Thus, as required,

$$I(a,\,b) = \left(\frac{b}{a+1}\right) I(a,\,b-1) \qquad \text{if } b > 0.$$ [**5 marks**]

It follows that $I(a,\,n) = n!/(a+1)^{n+1}$ and in particular

$$I(5,\,4) = \frac{4}{6} I(5,\,3) = \frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{1}{6}\int_0^1 x^5\,dx = \frac{24}{6^5} = \frac{4}{6^4} = \frac{1}{324}.$$ [**2 marks**]

(c) Substituting $x = y^t$ where $t > 0$, so that the limits are unchanged, we have

$$I(a,\,b) = \int_0^1 y^{ta}(-t\log y)^b t y^{t-1}\,dy = t^{b+1}\int_0^1 y^{ta+t-1}(-\log y)^b\,dy = t^{b+1} I(ta+t-1,\,b).$$

So choosing $t = 1/(1+a)$,

$$I(a,\,b) = \frac{I(0,\,b)}{(a+1)^{b+1}}.$$ [**5 marks**]

(d) Substituting in part (a) $t = -\log x$, so that $dt/dx = -1/x = -e^t$, while $x = 0$ corresponds to $t = \infty$ and $x = 1$ to $t = 0$,

$$\int_0^\infty t^4 e^{-t}\,dt = \int_1^0 [-\log x]^4\,(-dx) = \int_0^1 [-\log x]^4\,(dx) = I(0,\,4).$$ [**3 marks**]

Thus part (a) states that $I(0,\,4) = 24$, while from part (c)

$$I(5,\,4) = \frac{I(0,\,4)}{6^5} = \frac{24}{6^5} = \frac{1}{324}$$ [**2 marks**]

in agreement with part (b).

[**Total 20**]