Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The function $\coth x$ is defined by

$$\coth x = \frac{\cosh x}{\sinh x}.$$

(a) Express $\coth x$ in terms of the exponential function, and obtain an expression for the inverse function, $\coth^{-1} x$ in terms of the logarithmic function.

(b) Sketch the graph of $y = \coth^{-1} x$.

(c) If $y = \coth x$, express y' in terms of y and show that $y'' = 2y^3 - 2y$.

(d) Hence or otherwise, find the first three terms in the power series expansion of $\coth x$ about $x = a$ where $\coth a = 2$.

(e) State the complex value(s) of x for which $\coth x = 0$. (You need not prove that these are the only values,)

(f) Show that $\coth x \simeq 1 + 2e^{-2x}$ as $x \to \infty$ and that $\coth t \simeq (1/t)$ as $t \to 0$. Hence find the limit

$$\lim_{x\to\infty} \left[\frac{\log\left(\log\sqrt{\coth x}\right)}{\coth(1/x)} \right].$$

(g) Discuss whether or not the integral

$$\int_0^\infty [\coth(\coth x) - \coth 1]\, dx \qquad \text{exists.}$$

[*Do not try to evaluate the integral.*]

2. Write down an expression for the n^{th} derivative of $(x + c)^{-1}$ where c is a (possibly complex) constant.

Express the function

$$f(x) = \frac{2\sin\alpha}{x^2 - 2x\cos\alpha + 1}$$

in partial fractions, where α is a real constant, and deduce an expression for the n^{th} derivative evaluated at zero, $f^{(n)}(0)$. Hence show that

$$f(x) = \sum_{n=0}^{\infty} 2\sin[(n+1)\alpha]\, x^n.$$

Deduce that provided the series converges,

$$\frac{1}{\sin\alpha} = \sum_{n=0}^{\infty} (\cos\alpha)^n \sin[(n+1)\alpha]. \qquad (*)$$

When $\alpha = \frac{1}{3}\pi$, show that the right-hand side of $(*)$ can be written as the sum of two geometric series, and hence verify that the equation holds.

3. (a) State the mean value theorem precisely, and use it to prove that if a continuous and differentiable function $f(x)$ is such that its derivative $f'(x) = 0$ for $a < x < b$, then $f(x)$ must be constant for $a \leqslant x \leqslant b$.

(b) The differentiable function $f(x)$ obeys the equation

$$f(x) = \int_a^x [f(t)]^2 \, dt + x$$

where a is a given constant. By differentiating this equation or otherwise, find the unique solution.

(c) Find the general solution of the ODE

$$y' = \sec x - y \tan x.$$

4. (a) Evaluate the integral

$$\int_0^\infty u^4 e^{-u} \, du.$$

(b) The function $I(a,\, b)$, where $a > -1$ and $b > -1$ are real numbers, is defined by

$$I(a,\, b) = \int_0^1 x^a \, [-\log x]^b \, dx. \qquad (\dagger)$$

Use integration by parts to show that provided $b > 0$,

$$I(a,\, b) = \left(\frac{b}{a+1}\right) I(a,\, b-1).$$

Hence evaluate $I(a,\, n)$ when n is a positive integer, and show that $I(5,\, 4) = \frac{1}{324}$.

(c) Substituting $x = y^t$ where $t > 0$ in the integral in ($\dagger$), obtain a relation between $I(a,\, b)$ and $I(at + t - 1,\, b)$. Deduce that

$$I(a,\, b) = \frac{I(0,\, b)}{(a+1)^{b+1}}.$$

(d) Explain how part (a) is connected to parts (b) and (c) and show that your answers to (a), (b) and (c) are consistent.