UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2009

M1M1

Mathematical Methods 1

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2009

This paper is also taken for the relevant examination for the Associateship.

M1M1

Mathematical Methods 1

Date: examdate Time: examtime

All questions carry equal marks.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The function $f(x)$ is defined for all real x by

$$f(x) = \begin{cases} e^{-1/x} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0. \end{cases}$$

(a) Consider $\lim_{x \to 0} f(x)$. Is $f(x)$ continuous?

(b) Write down the Taylor series for $f(x)$ about the point $x = 1$ in terms of f and its derivatives, including the general term.

(c) Show that $x^2 f'(x) = f(x)$ and deduce that if $f^{(n)}$ denotes the n'th derivative of f,

$$f^{(n+1)}(1) + (2n-1)f^{(n)}(1) + n(n-1)f^{(n-1)}(1) = 0. \qquad (*)$$

(d) Assuming that for some real number L

$$\lim_{n \to \infty} \left[\frac{f^{(n+1)}(1)}{n f^{(n)}(1)} \right] = L,$$

deduce from equation $(*)$ the value of L.

(e) What is the radius of convergence of the series in part (b)?

Now define $g(t) = f(t^2)$, for all values of t.

(f) Assuming that $g^{(n)}(t)$ is continuous, and that $\lim_{y \to \infty} y^{\alpha} e^{-y} = 0$ for every real constant α, determine $g^{(n)}(0)$.

(g) Discuss carefully whether or not the **Maclaurin series** for $g(t)$ converges to the **function** $g(t)$.

(h) Does the integral

$$\int_0^{\infty} g(\log x)\, dx \qquad \text{exist?}$$

2. The complex variable $\zeta = \xi + i\eta$ is related to the complex variable $z = x + iy$ by

$$\zeta = \frac{e^z - i}{e^z + i}.$$

(a) Find $|\zeta|^2$ as a function of x and y and deduce that $|\zeta| < 1$ if $0 < y < \pi$.
What happens to the strip in the z-plane $\pi < y < 2\pi$?

(b) Illustrate in a diagram in the ζ-plane the curves corresponding to the lines $y = 0$, $y = \frac{1}{2}\pi$ and $y = \pi$, indicating the direction corresponding to increasing x.

(c) Find z as a function of ζ. What values of x and y correspond to the point $\zeta = (1+2i)/5$?

3. Find $y(x)$, implicitly or explicitly, satisfying the following ODEs and boundary conditions:

(a)

$$\cos x \frac{dy}{dx} + y \sin x = 1, \qquad y(\tfrac{1}{4}\pi) = 0.$$

(b)

$$y^2 = x^2 \left(\frac{dy}{dx} + 2 \right), \qquad y(1) = \tfrac{1}{2}.$$

(c)

$$y \frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 = 2 \frac{dy}{dx}, \qquad y(0) = 1, \quad y'(0) = 2,$$

by considering $(y^n y')'$ or otherwise.

4. (a) Find the limit

$$\lim_{x \to 2} (3 - x)^{\operatorname{cosec}(x-2)}.$$

(b) Simplify the expression

$$(a - b) \sum_{k=0}^{n-1} a^{n-k-1} b^k,$$

where n is an integer.

Hence find the derivative of $x^{1/n}$ **from first principles.**

(c) Write the real function $f(x) = \log[\sin(x + \pi/4)]$ as the sum of an even function $g(x)$ and an odd function $h(x)$.

Find the domain of x such that $-\pi < x < \pi$ for which all three of f, g and h are defined.

Sketch the graphs of f, g and h for $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$ on the same diagram.