1. (a) State precisely what it means for a function $f(x)$ to be differentiable at $x = a$. If $f(x)$ and $g(x)$ are differentiable at $x = a$, prove **from first principles** that $(fg)' = f'g + fg'$.

(b) Evaluate the limit:

$$\lim_{x \to 0} \frac{\log(1 + \sin x) - x}{\exp(x^2) - \cos x}.$$

(c) Find an even function $f(x)$ and an odd function $g(x)$ such that

$$f(x) + g(x) = 2^x.$$

Show that $f'(1) = (\log 2) g(1)$ and hence or otherwise show that

$$\lim_{x \to 1} \left(\frac{g(x-1)}{f(x) - f(1)} \right) = \frac{4}{3}.$$

2. (a) Find a formula for the n^{th} derivative of the function $f(z) = \log(z + \frac{1}{2})$. Hence derive a series expansion of $f(z)$ about $z = 0$, including the general term.

State the ratio test carefully, and derive the radius of convergence of this series.

(b) If $f(z)$ is as above and x is a real number, show that

$$\Re e\left[f(\tfrac{1}{2}e^{2ix})\right] = \log|\cos x|$$

where $\Re e$ denotes the real part, and find the corresponding imaginary part.

(c) Assuming the series expansion of part (a) is valid when $z = \frac{1}{2}e^{2ix}$, obtain the relation

$$x = \sin 2x - \tfrac{1}{2}\sin 4x + \tfrac{1}{3}\sin 6x + \ldots (-1)^{(n-1)}\frac{\sin 2nx}{n} + \ldots$$

Comment on the behaviour of this expression when $x = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ and $\frac{3}{4}\pi$.

3. (a) Sketch the graphs (i) $y = \tanh^{-1}(x)$, (ii) $y = \tanh(x^{-1})$ and (iii) $y = (\tanh x)^{-1}$.

(b) The differentiable function $f(x)$ is defined for $x \geqslant 0$ and is such that $f(x) > 0$ and $f(x) \to 0$ as $x \to \infty$.

The functions g and h are then defined by

$$g(x) = f(x) f\left(\frac{1}{x}\right) \qquad \text{and} \qquad h(x) = \frac{x f'(x)}{f(x)}.$$

Show that

$$\frac{x g'(x)}{g(x)} = h(x) - h\left(\frac{1}{x}\right).$$

If $h(x)$ is strictly decreasing ($h' < 0$), show that $g(x)$ has precisely one stationary point in $x > 0$, and find the value of x at which this occurs. By considering the sign of g' determine the nature of this stationary point.

Hence give a rough sketch of the curve $y = g(x)$ in $x > 0$.

4. (a) Use the substitution $t = \tan x$ to evaluate the integral

$$I = \int_0^{\pi/4} \frac{dx}{\varepsilon + \tan x} \qquad \text{where} \quad 0 < \varepsilon \ll 1.$$

Show that as $\varepsilon \to 0$

$$I = -\log \varepsilon - \tfrac{1}{2} \log 2 + O(\varepsilon).$$

(b) An approximation to the integral is obtained as follows. Introduce a number c where $0 < \varepsilon \ll c \ll 1$. Then

$$I \simeq \int_0^c \frac{dx}{\varepsilon + x} + \int_c^{\pi/4} \frac{dx}{\tan x}$$

Justify the approximations made, and evaluate the resulting integrals. Show that the two methods give approximately the same result when $\varepsilon \ll 1$.

5. (a) The shape of a long icicle hanging from the ceiling is given by the function $R(z)$, where R is the radius of the circular cross-section at height z above the ground. It can be shown that in terms of an angle ψ, that

$$\frac{dR}{dz} = \tan\psi \qquad \text{and} \qquad z = \frac{1}{\sin^4\psi}$$

Given that $R = 0$ at $z = 1$, show that

$$R = \tfrac{4}{3}(\sqrt{z} - 1)^{1/2}(2 + \sqrt{z}).$$

(b) Find $y(x)$ satisfying

$$\frac{dy}{dx} + e^x y = e^x \qquad \text{with} \qquad y(0) = 0.$$

(c) Find $y(x)$ satisfying

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$