1. Define the function

$$f(x) = \frac{1}{x}\left(\sqrt{x^2+1} - 1\right).$$

where the positive square root is assumed.

(a) Find the first three non-zero terms in the Taylor series expansion of this function about $x = 0$;

(b) Show that, as $x \to +\infty$,

$$f(x) \to 1 - \frac{1}{x} + \frac{1}{2x^2} + \ldots$$

(c) Does $f(x)$ have any stationary points in the domain $x > 0$?

(d) Sketch a graph of $f(x)$ for $x > 0$.

2. (a) Let $y(x) = \tan^{-1} x$. It is well-known that

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

By using this result, show that

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$

(b) Use the Leibniz rule to take the n-th derivative of the ordinary differential equation in part (a) and hence show that

$$y^{(n+2)}(0) = -n(n+1)y^{(n)}(0).$$

where $y^{(n)}$ denotes the n-th derivative of $y(x)$ with respect to x.

(c) Using the result from part (b), find the complete form of the sum representing the Taylor series of $\tan^{-1} x$ about $x = 0$.

3. (a) By considering the fact that, for all integers n,

$$\left(\cos\theta + i\sin\theta\right)^n = e^{in\theta},$$

find an expression for $\cos 4\theta$ as a polynomial in $\cos\theta$.

(b) By considering the series expansion of $\log(1-z)$ with the value $z = \frac{1}{2}e^{i\theta}$ where θ is real, show that

$$\log\left(\frac{2}{\sqrt{5 - 4\cos\theta}}\right) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n2^n}.$$

(c) Find all complex solutions of the equation

$$e^z + 2e^{-z} = 3.$$

4. (a) Find the following indefinite integrals:

$$(i) \int x \tan^{-1} x \; dx;$$

$$(ii) \int \frac{e^x + 1}{e^x - 1} dx;$$

$$(iii) \int \sqrt{x^2 + 2} \; dx.$$

(b) Define

$$I_n = \int_0^{\pi/2} \sin^{2n} x \; dx.$$

Show that

$$I_n = \left(\frac{2n - 1}{2n} \right) I_{n-1} \quad \text{for } n \geq 1.$$

Hence, show that

$$I_n = \frac{\pi (2n)!}{2^{2n+1} (n!)^2}.$$

5. (a) Find the general solution of the equation

$$x\frac{dy}{dx} = 1 + y^2.$$

(b) Find the solution of

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = x^3$$

satisfying the conditions $y(1) = 1, y'(1) = 0$.

(c) Find the general solution of

$$\frac{dy}{dx} = \frac{1}{x + y^2}.$$