1. The real function $y(x)$ is given by $y = \exp(\sin^{-1}(x))$.

(a) What is the largest domain over which this definition of $y(x)$ can apply?

(b) What is the maximum value attained by y for $0 \leqslant x \leqslant \frac{1}{2}$?

(c) Show that

$$(1 - x^2)y'' - xy' - y = 0. \qquad (*)$$

(d) If $y^{(n)}(x)$ denotes the n^{th} derivative of $y(x)$, obtain a relation between $y^{(n+2)}(0)$ and lower derivatives.

(e) Hence calculate terms up to x^5 in the power series for $y(x)$ about $x = 0$.

(f) Use part (d) to discuss the radius of convergence of the infinite power series in part (e), explaining any required extension to the usual arguments.

(g) Substituting $x = \cos\theta$ and defining $f(\theta) = y(x)$, derive a differential equation for $f(\theta)$ from $(*)$.

2. (a) Find the length of the cycloid curve defined by

$$x = t + \sin t \qquad y = 1 - \cos t \qquad \text{between } t = 0 \text{ and } t = \pi,$$

Sketch the curve over this range, paying attention to its gradient.

(b) Use the Mean Value Theorem to show that for $x > 0$

$$x > \tan^{-1} x > \frac{x}{1 + x^2}.$$

By integrating this inequality, deduce that

$$\tfrac{1}{2}\left(x + \frac{\log(1 + x^2)}{x}\right) > \tan^{-1} x > \frac{\log(1 + x^2)}{x}.$$

Solutions

1. (a) $\sin^{-1} x$ is only defined for $|x| \leqslant 1$, so the maximum domain is $|x| \leqslant 1$ **[1 mark]**

(b) $y' = \exp(\sin^{-1} x)/\sqrt{1-x^2} > 0$ for $|x| < 1$. Thus $y(x)$ is increasing and takes its maximum value for $0 \leqslant x \leqslant \frac{1}{2}$ at $x = \frac{1}{2}$. So the maximum value is $\exp[\sin^{-1}(1/2)] = \exp(\pi/6)$. **[2 marks]**

(c) We have $y'\sqrt{1-x^2} = y$ and so differentiating again

$$y''\sqrt{1-x^2} + \frac{y'(-x)}{\sqrt{1-x^2}} = y' \qquad \Longrightarrow \qquad (1-x^2)y'' - xy' = y'\sqrt{1-x^2} = y$$

[3 marks]

(d) Differentiating n times by Leibniz, we have

$$(1-x^2)y^{(n+2)} + n(-2x)y^{(n+1)} + \tfrac{1}{2}n(n-1)(-2)y^{(n)} - xy^{(n+1)} - ny^{(n)} = y^{(n)}.$$

Or

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2+1)y^{(n)} = 0.$$

Thus when $x = 0$,

$$y^{(n+2)}(0) = (n^2+1)y^{(n)}(0)$$ **[3 marks]**

(e) We have $y(0) = \exp(0) = 1$, $y'(0) = 1/1 = 1$. Using the above formula when $n = 0, 1, 2, 3$ then gives $y''(0) = y(0) = 1$,

$$y'''(0) = 2y'(0) = 2, \quad y''''(0) = 5y''(0) = 5, \quad y'''''(0) = 10y'''(0) = 20.$$

So from the Taylor/Maclaurin series

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = 1 + x + \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 + \tfrac{5}{24}x^4 + \tfrac{20}{120}x^5 \ldots$$

or

$$y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{5}{24}x^4 + \tfrac{1}{6}x^5 \ldots$$ **[4 marks]**

(f) If we consider the even and odd terms of the series separately, then the ratio of adjacent terms in each is

$$\left| \frac{y^{(n+2)}(0)x^{n+2}/(n+2)!}{y^{(n)}(0)x^n/n!} \right| = \frac{(n^2+1)x^2}{(n+2)(n+1)} \to x^2 \qquad \text{as } n \to \infty.$$

This is less than 1, provided $|x| < 1$. Thus the series consisting of the odd terms has a radius of convergence 1, as does the series consisting of the even terms. So we infer that the entire series also has Radius of Convergence 1. **[3 marks]**
It is also possible, but more cumbersome, to examine the ratio of adjacent terms.

(g) If $x = \cos\theta$ then $dy/dx = (dy/d\theta)(d\theta/dx) = f'(\theta)/(-\sin\theta)$. Differentiating again,

$$y'' = \frac{d\theta}{dx}\frac{d}{d\theta}\left[\frac{-f'}{\sin\theta}\right] = \frac{-1}{\sin\theta}\left[\frac{-f''}{\sin\theta} + \frac{f'\cos\theta}{\sin^2\theta}\right]$$

So Substituting in $(*)$ using $\sin^2\theta = (1-\cos^2\theta)$, we have

$$(f'' - f'\cot\theta) - \cos\theta\left(\frac{-f'}{\sin\theta}\right) - f = 0 \qquad \Longrightarrow \qquad f'' - f = 0. \qquad \textbf{[4 marks]}$$

2. (a) The required length is

$$L = \int_0^\pi \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt = \int_0^\pi [(1+\cos t)^2 + \sin^2 t]^{1/2} dt = \int_0^\pi \sqrt{2+2\cos t}\, dt$$

Now $1+\cos t = 2\cos^2\frac{1}{2}t$, so that as $\cos\frac{1}{2}t > 0$ in the integral range,

$$L = 2\int_0^\pi \cos\tfrac{1}{2}t\, dt = 2\left[2\sin\tfrac{1}{2}t\right]_0^\pi = 4. \qquad \textbf{[6 marks]}$$

Now $dy/dx = \sin t/(1+\cos t) = \tan\frac{1}{2}t$. This is always positive for $0 < t < \pi$ and approaches infinity as $t \to \pi$, indicating a vertical curve. The curve starts at the origin and increases towads $(\pi, 2)$, as in the figure. **[4 marks]**

(b) The MVT for the function $f(x) = \tan^{-1} x$ states that there exists a ξ in $0 < \xi < x$ such that

$$\frac{\tan^{-1} x - 0}{x - 0} = f'(\xi) = \frac{1}{1+\xi^2}.$$

Now as $1/(1+x^2)$ is a decreasing function, we have $1 > 1/(1+\xi^2) > 1/(1+x^2)$. It follows that, as required,

$$x > \tan^{-1} x > \frac{x}{1+x^2}. \qquad \textbf{[4 marks]}$$

It is legitimate to integrate inequalities, so that

$$\int_0^t x\, dx > \int_0^t \tan^{-1} x\, dx > \int_0^t \frac{x}{1+x^2}\, dx.$$

Or

$$\tfrac{1}{2}t^2 > \left[x\tan^{-1} x\right]_0^t - \int_0^t \frac{x\, dx}{1+x^2} > \left[\tfrac{1}{2}\log(1+x^2)\right]_0^t$$

Or

$$\tfrac{1}{2}t^2 > t\tan^{-1} t - \tfrac{1}{2}\log(1+t^2) > \tfrac{1}{2}\log(1+t^2).$$

Rearranging, we obtain

$$\tfrac{1}{2}\left(t + \frac{\log(1+t^2)}{t}\right) > \tan^{-1} t > \frac{\log(1+t^2)}{t}, \qquad \textbf{[6 marks]}$$

and replacing t by x gives the required answer.