M1M1: The Riemann Integral and the Fundamental Theorem of Calculus

This sheet can be found on http://www.ma.ic.ac.uk/∼ajm8/M1M1

Given an interval $[a, b]$ we define a partition to be a set of n points $x_1, x_2 \dots x_n$ such that

$$a \equiv x_0 < x_1 < x_2 < x_3 < \dots < x_n < x_{n+1} \equiv b.$$

For a given partition, we choose points on each subinterval, ξ_0, $\xi_1 \dots \xi_n$ such that for all i, $x_i < \xi_i < x_{i+1}$. Then for each function $f(x)$, we define the **Riemann sum**

$$S_n = (x_1 - x_0)f(\xi_0) + (x_2 - x_1)f(\xi_1) + \dots + (x_{n+1} - x_n)f(\xi_n).$$

Pictorially, we are forming $n + 1$ rectangles whose sum resembles the area under the curve $y = f(x)$.

We now let $n \to \infty$, in such a manner that $(x_{i+1} - x_i) \to 0$, for all i. If the sequence S_n tends to a limit, and if this limit does not depend on the particular partitions nor the values of ξ_i we choose, then we write this as

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$$

and the result is called the **integral** or **definite integral** of $f(x)$ between $x = a$ and $x = b$. A function $f(x)$ for which this limit exists is called **integrable**, or 'Riemann integrable.' It can be shown that all continuous functions are integrable. The function $f(x)$ is called the **integrand**.

It is important to grasp that an integral is a generalisation of a sum, and behaves similarly. Various properties follow from the definition. For example, if $f(x)$ is integrable and bounded by $m \leqslant f(x) \leqslant M$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a).$$

Suppose m and M are the minimum and maximum values attained by a continuous function $f(x)$ over the interval $[a, b]$. Then $(b - a)f(x)$ attains every value between $(b - a)m$ and $(b-a)M$ somewhere in $[a, b]$, in particular, the value equal to the above integral. Therefore there is a value ξ, in $a < \xi < b$ such that

$$\int_a^b f(x)\,dx = (b - a)f(\xi) \qquad \textbf{[The mean value theorem for integrals.]}$$

It also follows from the definition that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

If we define the integral from b to a in an identical manner we can see that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

The Fundamental Theorem of Calculus:

If we fix the lower limit a, then an integrable function $f(x)$ defines another function

$$F(b) = \int_a^b f(x)\,dx \qquad \text{or} \quad F(x) = \int_a^x f(t)\,dt.$$

Note that we do not use the same variable name for the limit (x) and the dummy integration variable, t, when there is any risk of confusion. It follows that for any c and d,

$$\int_c^d f(t)\,dt = \int_a^d f(t)\,dt - \int_a^c f(t)\,dt = F(d) - F(c).$$

Consider now

$$\int_x^{x+h} f(t)\,dt = F(x+h) - F(x).$$

By the Mean Value Theorem for Integrals (see above),

$$F(x+h) - F(x) = (x+h-x)f(\xi), \qquad \text{for some } \xi \text{ in} \quad x < \xi < x+h.$$

Thus

$$\lim_{h\to 0}\left[\frac{F(x+h)-F(x)}{h}\right] = \lim_{h\to 0} f(\xi).$$

But ξ is sandwiched between x and $x+h$, and so as $h \to 0$, necessarily $\xi \to x$. Thus the limit on the RHS exists and equals $f(x)$, as f is continuous. Hence the limit of the LHS exists. By definition, this means that the function $F(x)$ is differentiable, and its derivative is $f(x)$. We have shown that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad \textbf{[The Fundamental Theorem of Calculus.]}$$

Suppose now that $G(x)$ is another function such that $dG/dx = f(x)$, that is $G(x)$ is an **antiderivative** of f. Then

$$0 = f(x) - f(x) = \frac{dG}{dx} - \frac{dF}{dx} = \frac{d}{dx}(G-F) \qquad \Longrightarrow \qquad G - F = A \quad \text{for some constant } A.$$

Thus

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(b) + A - (F(a) + A) = G(b) - G(a).$$

So if $f(x)$ is a continuous function, we can evaluate its integral by taking the difference of any anti-derivative at the end-points. We write the anti-derivative as an **indefinite integral**

$$G(x) = \int^x f(t)\,dt \qquad \text{or} \quad G(x) = \int f(x)\,dx.$$