M1M1 Handout 1: Properties of the Trigonometric Power Series

This sheet can be found on http://www.ma.ic.ac.uk/∼ajm8/M1M1

Proof that $\cos x$ has a zero in the interval $(1.4, 1.6)$.

Let us denote by $\text{coz}\, x$ and $\text{zin}\, x$ the functions defined by the infinite power series. We aim to show that these are in fact the trigonometric functions $\cos x$ and $\sin x$. On this sheet and on Problem sheet 2 Q1, we prove various properties about $\text{coz}\, x$.

Firstly, we rearrange the power series for $\text{coz}\, x$ in the two equivalent forms:

$$\text{coz}\, x = \left[1 - \frac{x^2}{2}\right] + \frac{x^4}{4!}\left(1 - \frac{x^2}{(5)(6)}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{(9)(10)}\right) + \dots \tag{1}$$

and alternatively

$$\text{coz}\, x = \frac{1}{4!}\left[24 - 12x^2 + x^4\right] - \frac{x^6}{6!}\left(1 - \frac{x^2}{(7)(8)}\right) - \frac{x^{10}}{10!}\left(1 - \frac{x^2}{(11)(12)}\right) + \dots \tag{2}$$

Now from equation (1), we can see that $\text{coz}\, x > 0$, if

$$0 < x < \sqrt{2} \simeq 1.414,$$

since then all the bracketed terms are positive.

Now consider equation (2). The first term is a quadratic in x^2, and we can show that $24 - 12x^2 + x^4 < 0$ provided

$$6 + \sqrt{12} > x^2 > (6 - \sqrt{12}).$$

We note that $(6 - \sqrt{12})^{1/2} \simeq 1.592$. Furthermore, all the remaining terms are negative provided $x^2 < 56$. This is certainly true if $x^2 < 6 + \sqrt{12}$. Thus we know $\text{coz}\, x < 0$ in the above range.

So $\text{coz}\, x$ changes sign somewhere between $x = 1.414$ and $x = 1.592$. Since $\text{coz}\, x$ is a continuous function, we conclude there must be a value τ for which $\text{coz}\, \tau = 0$.

Of course we 'know' that $\tau = \frac{1}{2}\pi \simeq 1.57$. We will in fact define π as equal to 2τ, where τ is the first positive zero of $\text{coz}\, x$.

Exercise: Use the power series for zin to prove that $\text{zin}\, x > 0$ for $0 < x < \sqrt{6}$. Deduce that $\sin \tau > 0$.

You may need this last result in question 1 on problem sheet 2, where you will prove that $\text{coz}\,(x + \tau) = -\text{zin}\, x$, and that coz is a 4τ-periodic function. You will also need the formula for $\text{coz}\,(x + y)$ which we prove next.

[**Note:** We are assuming that the infinite series **converge** which we haven't yet shown. We shall also assume overleaf that we can rearrange all the terms in the product of two infinite series, which you will prove next term.]

Proof of the formula for $\cos(x + y)$

We begin by arguing that a double sum over all positive integers m and n is equivalent to summing over all possible totals $p \equiv m + n$ and over each value of n less than this total, i.e.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[\ldots] = \sum_{p=0}^{\infty}\sum_{n=0}^{p}[\ldots].$$

Now from the series definition of the coz function, (being careful to use different dummy summation variables, m and n)

$$\begin{aligned}
\operatorname{coz} x \operatorname{coz} y &= \left[\sum_{n=0}^{\infty}\frac{(-1)^n\, x^{2n}}{(2n)!}\right]\left[\sum_{m=0}^{\infty}\frac{(-1)^m\, y^{2m}}{(2m)!}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+m}\, x^{2n}\, y^{2m}}{(2n)!(2m)!}\right] \\
&= \sum_{p=0}^{\infty}\sum_{n=0}^{p}\frac{(-1)^p\, x^{2n}\, y^{2p-2n}}{(2n)!(2p-2n)!} \\
&= \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p)!}\sum_{n=0}^{p}\binom{2p}{2n}x^{2n}y^{2p-2n}
\end{aligned} \tag{3}$$

using the double sum relabelling above, writing $p = (m + n)$ and using the definition of the binomial coefficient. Similarly,

$$\begin{aligned}
-\operatorname{zin} x \operatorname{zin} y &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}+\frac{(-1)^{m+n+1}x^{2n+1}y^{2m+1}}{(2n+1)!(2m+1)!} \\
&= \sum_{p=1}^{\infty}\frac{(-1)^p}{(2p)!}\sum_{n=0}^{p}\binom{2p}{2n+1}x^{2n+1}y^{2p-(2n+1)}
\end{aligned} \tag{4}$$

where this time we have written $p = (m + n + 1)$. Note that m, n and p are dummy variables we sum over and they hold no significance outside the equation they appear in.

The RHSs of equations (3) and (4) are very similar – the first involves all the even integers up to $2p$ while the second all odd integers. Adding them together, we have

$$\begin{aligned}
\operatorname{coz} x \operatorname{coz} y - \operatorname{zin} x \operatorname{zin} y &= \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p)!}\sum_{k=0}^{2p}\binom{2p}{k}x^k y^{2p-k} \\
&= \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p)!}(x+y)^{2p} \qquad \text{by the binomial theorem} \\
&= \operatorname{coz}(x+y)
\end{aligned}$$

by the definition of coz . We have proved that

For the functions coz and zin defined by infinite series, then for all x and y $\operatorname{coz}(x+y) = \operatorname{coz} x \operatorname{coz} y - \operatorname{zin} x \operatorname{zin} y$.