Name (IN CAPITAL LETTERS!): .... **TID:**

CID: .... **Personal tutor:** ....

Question 3. "Zweibak", "Bones" and "Doubting Thomas" share a house at $x = 1$. To help them split their weekly grocery bill, find the constants a_n such that

$$\frac{1}{x^2 - 2x + 3} = \sum_{n=0}^{\infty} a_n (x-1)^n. \qquad [\mathbf{4}]$$

"Doubting Thomas" doubts this formula is fair if he wanders too far from home. Determine all values of x for which the series converges. [**4**]
What feature of the original function is responsible for the series diverging for other values of x? [**2**]
*Note: numbers like [**2**] indicate the number of marks available for each portion.*

Answer. Write $x - 1 = t$. Then

$$\frac{1}{x^2 - 2x + 3} = \frac{1}{2 + t^2} = \tfrac{1}{2}(1 + t^2/2)^{-1} = \tfrac{1}{2}(1 - t^2/2 + t^4/4 + \ldots) = \tfrac{1}{2}\sum_{k=0}^{\infty}(-1)^k(\tfrac{1}{2}t^2)^k$$

It follows that $a_n = 0$ if n is odd, while if $n = 2k$, then $a_{2k} = (-1)^k/2^{k+1}$ **(4 marks)**.
Using the ratio test, we consider

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}/2^{k+2}t^{2k+2}}{(-1)^k/2^{k+1}t^{2k}}\right| = |t^2|/2.$$

By the ratio test, the series converges if $|(x-1)^2| < 2$, so that $1 - \sqrt{2} < x < 1 + \sqrt{2}$. If $t^2 = 2$, then the series is $\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots$ which obviously does not converge. (Although it oscillates about the "correct" value 1/4.) So the series converges if and only if $1 - \sqrt{2} < x < 1 + \sqrt{2}$. **(4 marks)**

The function is singular at the points in the complex plane, $x - 1 = \pm\sqrt{2}i$, which restricts the radius of convergence to $\sqrt{2}$, even for real x. **(2 marks)**

Note to markers: As ever, feel free to use your discretion. Deduct 1 if they neglect to state that they take a limit as $n \to \infty$ in the ratio test. Deduct 1, for not considering the convergence for $x = 1 \pm \sqrt{2}$.