Name (IN CAPITAL LETTERS!): ....

CID: ....

Question 4. Hairdressing in Huxley requires considerable mathematical expertise. When Mr T. asks his barber to trim his beard to the shape $y = f(x)$, this should be interpreted as a request for an even cut and so the odd part of Mr. T.'s request should be ignored. One week, the barber therefore expresses

$$f(x) = \frac{x}{1-x} \equiv g(x) + h(x),$$

where $g(x)$ is an even function of x and $h(x)$ is odd.

He then draws $f(x)$, $g(x)$ and $h(x)$ on the same diagram, carefully labelling each curve and highlighting any turning points and points of inflection in any of the 3 curves. By considering the signs of f, g and h he indicates clearly those regions where $f > g > h$, $f > h > g$ and so on.

Can you do as well as Mr. T's barber?

Answer. Using $g(x) = \frac{1}{2}[f(x) + f(-x)]$, we find

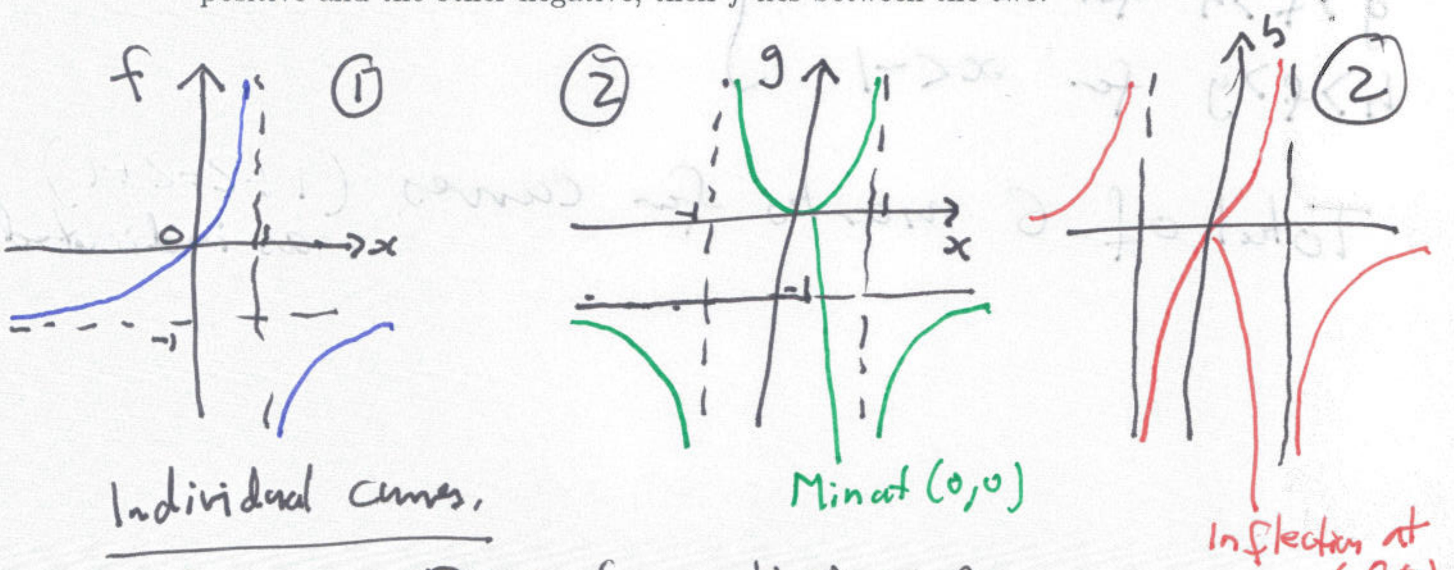
$$f(x) = \frac{x}{1-x}, \qquad g(x) = \frac{x^2}{1-x^2}, \qquad h(x) = \frac{x}{1-x^2}. \qquad \textbf{(2 marks)}$$

Now $f(x) = -1 + 1/(1-x)$, a rectangular hyperbola with asymptotes $x = 1$ and $y = -1$. The curve $y = f(x)$ has no turning points or points of inflection.

Similarly $g(x) = -1 + 1/(1-x^2)$. This has asymptotes $x = \pm 1$, $y = -1$. Differentiating $g' = 2x/(1-x^2)^2$ and $g'' = 2/(1-x^2)^2 + 8x^2/(1-x^2)^3 = (2 + 6x^2)/(1-x^2)^3$. Thus g has a minimum at $x = 0$, and no points of inflection. It is positive iff $|x| < 1$.

Now $h' = 1/(1-x^2) + 2x^2/(1-x^2)^2 = (1+x^2)/(1-x^2)^2$ and is always positive. $h'' = 2x/(1-x^2)^2 + 4x(1+x^2)/(1-x^2)^3 = 2x(3+x^2)/(1-x^2)^3$. So h' is minimum at $x = 0$, so this is a point of inflection.

Now if $|x| > 1$, then $|g| > |h|$, and the converse if $|x| < 1$. As $f = g + h$, if g and h have the same sign, f is larger in modulus than either. If one of h and g is positive and the other negative, then f lies between the two.



Individual curves.

PTO for combined curves