Name (IN CAPITAL LETTERS!): .... **TID:**

CID: .... **Personal tutor:**............................

Question 2. A function $f(x)$ can be differentiated any number of times for all x. It also satisfies the relationship

$$f'(x) - 2xf(x) = 0 \qquad \text{for all } x\text{, while } f(0) = 1.$$

Differentiating this equation $(n+1)$ times, show that for natural numbers n,

$$f^{(n+2)}(0) = (2n+2)f^{(n)}(0). \qquad [\mathbf{2}]$$

Find $f^{(n)}(0)$ if n is odd, [**2**] and use induction to show that if n is even $(n = 2r)$

$$\frac{f^{2r}(0)}{(2r)!} = \frac{1}{r!}. \qquad [\mathbf{4}]$$

Hence write down the Taylor series for $f(x)$ about $x = 0$. [**1**]

What is a simpler form of $f(x)$? [**1**]

Note: numbers like [**2**] *indicate the number of marks available for each portion.*
You may answer the later parts even if you have left out some of the earlier ones.

Answer. Differentiating $(n+1)$ times by Leibniz,

$$f^{(n+2)}(x) - 2xf^{(n+1)}(x) - 2(n+1)f^{(n)}(x) = 0 \qquad \Longrightarrow f^{(n+2)}(0) = (2n+2)f^{(n)}(0).$$

(2 marks)

Now $f(0) = 1$, and by the given equation $f'(0) = 0$. It follows $f^{(3)}(0) = 0$ and recursively, $f^{(n)}(0) = 0$ for all odd n. **(2 marks)**

Now if $r = 1$, $n = 2$ and $f''(0) = 2f(0) = 2$. So $f''(0)/2! = 1 = 1/1!$ so the result is true for $r = 1$. [Also ok to substitute $r = 0$.] **(1 mark)**

Assume true for $r = k$. Then

$$\frac{f^{(2(k+1)}(0)}{(2k+2)!} = \frac{(4k+2)f^{2k}(0)}{(2k+2)!} = \frac{2(2k+1)}{(2k+2)(2k+1)}\frac{f^{(2k}(0)}{2k!} = \frac{1}{k+1}\frac{1}{k!} = \frac{1}{(k+1)!}$$

So result is true for $k+1$, hence true for all r by induction. **(3 marks)**

Thus, the Taylor series is

$$f(x) = 1 + x^2 + \frac{x^4}{2!} + \ldots \frac{(x^2)^r}{r!} + \ldots \qquad \textbf{(1 mark)}$$

which we recognise as the series for $\exp(x^2)$. **(1 mark)**