**Name (IN CAPITAL LETTERS!):** ........................................

**CID:** ....................................................................

**Question 3.** The moon is a distance $d$ from the earth and, as every cultured person knows, $d$ is approximately $4 \times 10^{10}$cm. The mass of the earth is about 100 times that of the moon, due partly to the weight of the pure mathematicians.

At a distance $x$ from the earth, where $0 < x < d$, the spaceship "Greybeard" has a gravitational potential energy

$$V(x) = \frac{100}{x} + \frac{1}{(d-x)}.$$

(a) Find the point $x = m$ where the potential $V$ is minimum.

(b) Writing $y = x - m$, express the energy $V$ as a power series in $y$, and show that

$$V = \sum_{n=0}^{\infty} \frac{11^{n+1} y^n}{d^{n+1}} \left(1 + (-1)^n 10^{1-n}\right).$$

(c) Use the ratio test to find the radius of convergence of this series, and state the values of $x$ for which it converges.

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**Answer.** Setting $V'(m) = 0$, we have

$$\frac{100}{m^2} = \frac{1}{(d-m)^2} \quad \Longrightarrow \quad \frac{10}{m} = \pm\frac{1}{d-m} \quad \Longrightarrow \quad 10d = (10 \pm 1)m$$

As we must have $0 < m < d$, $m = \frac{10}{11}d$. As $V$ is large and positive as $x \to 0$ and as $x \to d$, and the least value must be attained somewhere, $x = m$ must be a minimum. **(2 marks)**

Now

$$V = \frac{100}{m+y} + \frac{1}{d-m-y} = \frac{100}{m}\left(1 + \frac{y}{m}\right)^{-1} + \frac{1}{d-m}\left(1 - \frac{y}{d-m}\right)^{-1}$$

Thus provided the series converge,

$$V = \frac{100}{m}\sum_{n=0}^{\infty}\frac{(-1)^n y^n}{m^n} + \frac{1}{d-m}\sum_{n=0}^{\infty}\left(\frac{y}{d-m}\right)^n = \sum_{n=0}^{\infty} y^n \left(\frac{100(-1)^n}{m^{n+1}} + \frac{1}{(d-m)^{n+1}}\right)$$

Or substituting $m = \frac{10}{11}d$,

$$V = \sum_{n=0}^{\infty} \frac{11^{n+1}y^n}{d^{n+1}} \left(1 + (-1)^n 10^{1-n}\right)$$

as required. **(4 marks)**

To find the radius of convergence, we use the ratio test. The ratio of the $n^{th}$ to the $(n-1)^{th}$ terms has the limit

$$\lim_{n\to\infty} \left| \frac{(11/d)^{n+1}y^n \left(1 + (-1)^n 10^{1-n}\right)}{(11/d)^n y^{n-1} \left(1 + (-1)^{n-1} 10^{2-n}\right)} \right| = 11|y|/d,$$

since for large $n$, $10^{-n}$ is very small. Thus the series converges provided $|y| < d/11$, **(3 marks)** or for $\frac{9}{11}d < x < d$. **(1 marks)**.

We note that $V$ has two singularities, at $y = \frac{1}{11}d$ and $y = -\frac{10}{11}d$. The radius of convergence is the shortest distance to a singularity.

The point $x = m$ is where the spaceship starts getting pulled towards the moon, rather than back to earth. We have ignored the rotation of the moon around the earth. Try Googling "Lagrange points" if interested.

*Note to markers: This may be tiresome to mark. Whenever an answer is given in the question, some people will just write it down whenever they get close to it, and hope you will credit them with having done the arithmetic in their head. It is up to you how much credence to give to this. For example, the solution I give above has a slightly suspicious jump to the answer, and you might be justified in not giving it full marks, if you choose. Personally, I am harsher if I estimate that the student him/herself does not believe the answer follows (i.e. they are LYING), but this is your call.*

*Some may quote the convergence criteria for the two series separately ($|y| < m$ and $|y| < d - m$), and infer that both must apply. Give some credit for this if presented clearly – again use your judgement.*