Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 1. A curve is given in terms of a parameter t, by

$$x = X(t), \qquad y = Y(t) \qquad \text{for all } t,$$

where the functions $X(t)$ and $Y(t)$ are given. Obtain an expression for the 2nd derivative, d^2y/dx^2, in terms of X, Y, and their derivatives.
The "cycloid" is defined for all t by

$$x = t - \sin t, \qquad y = 1 - \cos t.$$

Express dy/dx and d^2y/dx^2 in terms of t. For which values of t, if any, does the curve (a) have infinite gradient, (b) have inflection points?

Answer. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{Y'(t)}{X'(t)}$$

Thus, differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{Y'(t)}{X'(t)}\right)\frac{dt}{dx} = \frac{X'Y'' - Y'X''}{X'^3}$$ **(3 marks)**

For the given curve, $X' = 1 - \cos t$, $Y' = \sin t$, $X'' = \sin t$, $Y'' = \cos t$, and so

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$$ **(1 mark)**

This might be infinite when $\cos t = 1$, for example at $t = 0$. Near $t = 0$, using the power series, we see that

$$\frac{dy}{dx} \simeq \frac{t}{t^2/2} \to \infty \qquad \text{as } t \to 0.$$

As dy/dx is clearly 2π-periodic in t, we conclude the gradient is infinite for $t = 2n\pi$ for integer n. [No marks for not noticing the numerator vanishes.] **(2 marks)**

$$\frac{d^2y}{dx^2} = \frac{(1-\cos t)\cos t - \sin^2 t}{(1-\cos t)^3} = \frac{\cos t - 1}{(1-\cos t)^3} = \frac{-1}{(1-\cos t)^2}$$ **(2 marks)**

This is never zero, so we conclude the curve has no inflection points. **(2 marks)**