Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 4. (a) Mr T was so jealous of Mr L's popularity function in last week's test, that he decided to try to differentiate the (continuous) function

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \qquad \text{for } x \neq 0, \quad \text{and} \quad f(0) = 0.$$

(a) Using any method, help Mr T find $f'(x)$, for $x \neq 0$.

(b) For $x = 0$, show **from first principles** that the derivative $f'(0)$ exists and find it.

(c) If $g(x)$ is differentiable for all x, must the derivative, $g'(x)$ be continuous?

Answer. (a) We have for $x \neq 0$ using the product and chain rules

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \qquad [\mathbf{2}]$$

(b) From the definition of the derivative, $f'(0)$, if it exists, is given by the limit

$$f'(0) = \lim_{\varepsilon\to 0}\left(\frac{f(\varepsilon) - f(0)}{\varepsilon}\right) = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon^2\sin(1/\varepsilon) - 0}{\varepsilon}\right) = \lim_{\varepsilon\to 0}\left[\varepsilon\sin(1/\varepsilon)\right] = 0, \qquad [\mathbf{4}]$$

since $|\sin t| \leqslant 1$ for all t. We see therefore that $f'(0)$ exists and $f'(0) = 0$.

(c) Combining (a) and (b) we see that $f'(x)$ exists for all x. However, the formula in part (a) does not tend to a limit as $x \to 0$ as the cosine is undefined for $x = 0$. Therefore the derivative $f'(x)$ is discontinuous at $x = 0$ (but it exists for all x). So no, the derivative need not be continuous (choose $g(x) = f(x)$). [**4**]