Name (IN CAPITAL LETTERS!): ....

CID: ....

Question 1. The Starship Enterprise, under the able leadership of Chief Engineer Dr "Bones" Mccoy, is on a conical orbit around a strange new world. Its five-year mission is to boldly introduce new transponder technology into the Imperial Mathematics course.

Extensive study has shown that when asked a yes/no question by a lecturer, the proportion of the class who attempt an answer, N, depends on the difficulty of the question asked, x, and also on the time during the lecture. If x can take any positive real value, while the time t lies in $(0,1)$, then it is found that

$$N = \left(t(1-t) + \tfrac{3}{4}\right)\left(\frac{2x}{4+x^2} + \tfrac{1}{2}\right).$$

The number who enter a correct answer into their electronic gadget, C, depends on N and x according to the formula

$$C = \tfrac{1}{2}N\left(\frac{x^2+8}{(x+2)^2}\right).$$

(a) At what time during a lecture is the number of student responses highest?
[From now on you may if you wish assume the question is asked at this time.]
(b) What level of difficulty gives the maximum number of responses?
(c) What level of difficulty gives the minimum ratio of correct answers to number of attempts?
(d) What level of difficulty gives the maximum total of correct answers?

Answer.
(a) Whatever the value of x, N is maximised when $t(1-t) + \frac{3}{4}$ is maximum. This occurs when $t = \frac{1}{2}$. **(2 marks)** Then $t(1-t) + \frac{3}{4} = 1$, and may subsequently be ignored.

(b) Whatever the value of t, N is maximised when $2x/(4+x^2) + \frac{1}{2}$ is maximised. Differentiating,

$$\frac{(4+x^2)2 - 2x(2x)}{(4+x^2)^2} = 0 \Longrightarrow x = 2.$$

As N is clearly positive for $x > 0$, and is zero at $x = 0$ and also at $x = \infty$, this must be a maximum. **(2 marks)**

[We note everyone gets the answer right at $t = 1/2$ and $x = 2$, as $N = 1$ then.]

(c) The ratio C/N is 1 at $x = 0$ and $1/2$ as $x \to \infty$. Now

$$\frac{C}{N} = \tfrac{1}{2}\frac{(x+2)^2 - 4x + 4}{(x+2)^2} = \tfrac{1}{2}\left[1 - \frac{4}{x+2} + \frac{12}{(x+2)^2}\right]$$

Differentiating,

$$\left(\frac{C}{N}\right)' = \tfrac{1}{2}\left[\frac{4}{(x+2)^2} - \frac{24}{(x+2)^3}\right] = \frac{2x-8}{(x+2)^3}.$$

This is zero if and only if $x = 4$. When $x = 4$, $C/N = 1/3$, which is less than the values at $x = 0$ and as $x \to \infty$, and so is a minimum.**(3 marks)**

(d) After some simplification, we find that

$$C = \tfrac{1}{4}\frac{x^2 + 8}{4 + x^2} = \tfrac{1}{4}\left[1 + \frac{4}{4 + x^2}\right].$$

This is clearly a decreasing function of x, and so the maximum occurs when $x = 0$. **(3 marks)**

Most correct answers are obtained when the question is easiest, but clearly half the class can't be bothered answering unless the question is a little challenging.

Note to markers: I fear you will get some long-winded algebra, if they rush into differentiating quotients without thinking and simplifying first. I leave it up to you how much justification of the nature of the turning points should be required. I have given fairly brief common sense arguments above. Have fun!