Name (IN CAPITAL LETTERS!): .... **TID:** ☐

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Question 3. (a) Express the function $\exp(\cos x)$ as a power series in x, neglecting powers higher than x^4.

(b) The popularity of Mr L, Mr T and Ms M in their N^{th} lecture, varies according to the functions

$$L(N) = \frac{N^2\sin(1/N)}{N+\sin(N)}, \qquad T(N) = \frac{N^{100}}{(1.01)^N}, \qquad M(N) = N(\sqrt{N^2+1}-N).$$

Determine the behaviour of these functions as the number of lectures, $N \to \infty$.

Answer. When $x \simeq 0$, we know $\cos x \simeq 1$, and so we DO NOT expand $\exp(\cos x) = 1 + \cos x + \ldots\ldots$ as we cannot then neglect the high powers of $\cos x$. Instead, write

$$\exp(\cos x) = \exp(1+T) \qquad \text{where} \quad T = -x^2/2 + x^4/24 + \ldots$$

$$= \exp(1)\left[1 + T + \frac{T^2}{2} + \ldots\right] = e\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \tfrac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\ldots\right)^2 + \ldots\right]$$

$$= e\left[1 - \frac{x^2}{2} + x^4\left(\frac{1}{24} + \frac{1}{8}\right)\right] = e - \frac{e}{2}x^2 + \frac{e}{6}x^4 + O(x^6) \qquad \textbf{(4 marks)}$$

(b) (i) as $N \to \infty$, $\sin N$ is undefined; however $(\sin N)/N \to 0$. Also $\sin(1/N) = 1/N + O(1/N^3)$. Thus

$$\lim_{N\to\infty} L(N) = \lim_{N\to\infty}\left[\frac{N^2\sin(1/N)}{N+\sin N}\right] = \lim_{N\to\infty}\left[\frac{N+O(1/N)}{N(1+(\sin N)/N)}\right] = 1 \qquad \textbf{(2 marks)}$$

(ii) In any conflict, an exponential beats a power, so that (since $\log 1.01 > 0$)

$$\lim_{N\to\infty} T(N) = N^{100}e^{-\log(1.01)N} = 0. \qquad \textbf{(2 marks)}$$

(iii) **Either:** Using the binomial series

$$(N^2+1)^{1/2} = N(1+1/N^2)^{1/2} = N + 1/(2N) + \ldots$$

so that

$$\lim_{N\to\infty} M(N) = \lim_{N\to\infty}\left[N\left(N + \frac{1}{2N} - N\right)\right] = \tfrac{1}{2} \qquad \textbf{(2 marks)}$$

or multiply top and bottom by $\sqrt{N^2+1}+N$ and use the difference of two squares

$$\lim_{N\to\infty} M(N) = \lim_{N\to\infty}\left[\frac{N(1+N^2-N^2)}{\sqrt{N^2+1}+N}\right] = \lim_{N\to\infty}\left[\frac{N}{N+O(1/N)+N}\right] = \tfrac{1}{2}.$$