Name (IN CAPITAL LETTERS!): . . . .

CID: . . . .

Question 4. Using the series and methods described in lectures, express as a power series about $x = 0$ the function
$$\frac{1+\sin x}{1+\cos x}$$
including terms up to x^4.

Answer.

Writing $y = 1 - \cos x \equiv \frac{1}{2}x^2 - \frac{1}{24}x^4 + O(x^6)$,

$$\frac{1+\sin x}{1+\cos x} = \frac{1+\sin x}{2-y} = \tfrac{1}{2}(1+\sin x)(1-\tfrac{1}{2}y)^{-1}$$

$$= \tfrac{1}{2}(1+\sin x)\left(1+\tfrac{1}{2}y+(\tfrac{1}{2}y)^2+O(y^3)\right)$$
$$= \tfrac{1}{2}\left[1+x-\tfrac{1}{6}x^3+O(x^5)\right]\left[1+\tfrac{1}{4}x^2-\tfrac{1}{48}x^4+\tfrac{1}{4}(x^4/4)+O(x^6)\right]$$
$$= \tfrac{1}{2}\left[1+x-\tfrac{1}{6}x^3+\tfrac{1}{4}x^2+\tfrac{1}{4}x^3+\tfrac{1}{24}x^4+O(x^5)\right]$$
$$= \tfrac{1}{2}+\tfrac{1}{2}x+\tfrac{1}{8}x^2+\tfrac{1}{24}x^3+\tfrac{1}{48}x^4+O(x^5)$$

This test is about accuracy as well as method. I suggest awarding one mark for getting the sine and cosine series correct, two marks for expanding in y rather than $\cos x$ (which is NOT small near $x = 0$), one mark for getting the expansion in $1/(2-y)$ correct, amd then 1 mark for getting each of the 5 terms in the power series correct. You, the marker, may modify this scheme as you see fit. For example, if they forget the factor of a half at the front, you may wish only to deduct one mark. In general be more sympathetic to clear, well written mathematics than to random scrawls of symbols. Some may attempt other methods - they can try using trigonometric manipulations if they wish, but they may not use Taylor/MacLaurin series - penalise appropriately.