Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 2. The function Zin(x) is defined by the power series

$$\mathrm{Zin}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

An eminent pure mathematician, Thomas Richards, assures you that the infinite series converges for every value of x and that any sensible manipulation of the series is legitimate. Show that

(a) $\mathrm{Zin}(x) > 0$ for $0 < x < \sqrt{6}$.

(b) If $g(x) = \mathrm{Zin}(x) - \frac{1}{6}x$, then $g(\sqrt{10}) < 0$.

Answer. This exercise closely mirrors the calculation for the (more useful) cosine series on the handout shown in lectures which can be found on the Website.

(a) The series can be written

$$\mathrm{Zin}(x) = \left[x\left(1 - \frac{x^2}{6}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6(7)}\right) + \frac{x^9}{9!}\left(1 - \frac{x^2}{10(11)}\right) \dots \right]$$

Now if $x^2 < 6$, then $x^2 < 42$, $x^2 < 110\dots$ also, and hence all of the brackets are positive. Thus if $0 < x < \sqrt{6}$ all terms in the rearranged series are positive, and we deduce $\mathrm{Zin}(x) > 0$ if $0 < x < \sqrt{6}$. **(5 marks)**

(b) Now regroup as follows:

$$\mathrm{Zin}(x) - \frac{x}{6} = \left[\left(\frac{5x}{6} - \frac{x^3}{6} + \frac{x^5}{120}\right) - \frac{x^7}{7!}\left(1 - \frac{x^2}{8(9)}\right) - \frac{x^{11}}{11!}\left(1 - \frac{x^2}{12(13)}\right) \dots \right]$$

When $x = \sqrt{10}$ all brackets other than the first are positive, so that all terms other than the first are negative. The first bracket is $\frac{x}{120}(100 - 20x^2 + x^4) = \frac{x}{120}(10 - x^2)^2$ which is zero when $x = \sqrt{10}$. Thus

$$g(\sqrt{10}) = 0 - \dots - \dots - \dots < 0$$ **(5 marks)**

[Note to markers: You will get some sloppy attempts here, and may deduct marks for lack of clarity. Judge each script according to whether what is written would convince you of the result, and whether the writer understands the argument.]