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Question 1.
(a) The **odd** function $f(x)$ and the **even** function $g(x)$ are defined for all x. In addition, it is known that $f(x) = 0$ for precisely one value of x.

Determine whether the following three functions of x are necessarily even or odd or not necessarily either:

$$(i)\ \ f[g(x)] - g[(f(x)] \qquad (ii)\ \ \Big(f[f(x)]\Big)\Big(g[g(x)]\Big) \qquad (iii)\ \ \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{f[f(x)]} & \text{for } x \neq 0 \end{cases}$$

(b) Find the inverse function, h^{-1} together with its domain, given that

$$h(x) = \frac{x^2}{x^2+1}, \qquad \text{for} \quad x \geqslant 0.$$

Answer. (a)(i)

$$f(g(-x)) - g(f(-x)) = f(g(x)) - g(-f(x)) = f(g(x)) - g(f(x)).$$

So this function is even. **(2 marks)**

(ii) $\ f(f(-x)).g(g(-x)) = f(f(x)).g(-g(x)) = f(f(x).[-g(g(x))] = -f(f(x).[g(g(x))].$

So this function is odd. **(2 marks)**

(iii) As $f(x)$ is odd, $f(0) = 0$. As $f(x)$ is only zero for one value of x, $f(x) \neq 0$ if $x \neq 0$. Therefore $f(f(x)) = 0$ only if $f(x) = 0$, which only happens if $x = 0$. We conclude that the function, as given, is well defined. If $x \neq 0$, then

$$\frac{1}{f(f(-x))} = \frac{1}{f(-f(x))} = \frac{1}{-f(f(x))} = -\frac{1}{f(f(x))}$$

Bearing in mind that the function is zero when $x = 0$, we conclude this function is odd. [Deduct 1 if whether the denominator vanishes is not considered.] **(3 marks)**

(b) Putting

$$y = h(x) = \frac{x^2}{1+x^2} \quad \Longrightarrow \quad x^2 = \frac{y}{1-y} \quad \Longrightarrow \quad x = +\sqrt{\frac{y}{1-y}} = h^{-1}(y),$$

recalling that the domain of h is $x > 0$. (Deduct one if the reason for choosing the plus sign is not stated.) **(2 marks)**

The domain of h^{-1} is the range of $h(x) = 1 - 1/(1+x^2)$. This takes all values between 0 and 1, including 0 but excluding 1. Thus the domain of $h^{-1}(x)$ is $[0,1)$ (note brackets) or $0 \leqslant x < 1$. **(1 mark)**