Name (IN CAPITAL LETTERS!): ....

CID: ....

Question 2. In a perfect world, there would be an infinite number of M1F lectures. Experience shows that the difficulty of each 'clicker' question doubles each lecture, while the number of attempts in the n^{th} lecture is proportional to $\cos nx$, where x is the number of students and does not vary between lectures. The total number of correct answers received during this never-ending course is thus proportional to

$$S = \sum_{n=1}^{\infty} \frac{\cos nx}{2^n}.$$

By writing S as the real part of a complex series, show that

$$S = \frac{2\cos x - 1}{5 - 4\cos x}.$$

If S were expanded as a power series in x about $x = 0$, what would you expect the radius of convergence of that series to be? [Do not find this series.]

Answer.

$$S = \Re e \left[\sum_{n=1}^{\infty} \frac{\exp(inx)}{2^n} \right] = \Re e \left[\sum_{n=1}^{\infty} \left(\frac{\exp(ix)}{2} \right)^n \right] = \Re e \left[\frac{e^{ix}/2}{1 - e^{ix}/2} \right] = \Re e \left[\frac{e^{ix}}{2 - e^{ix}} \right],$$

(4 marks) assuming the series converges, so that $|e^{ix}| < 2$. Multiplying top and bottom by the complex conjugate of the denominator, $2 - \exp(-ix)$, we have as required

$$S = \Re e \left[\frac{2e^{ix} - 1}{4 - 2e^{ix} - 2e^{-ix} + 1} \right] = \frac{2\cos x - 1}{5 - 4\cos x} \qquad \textbf{(3 marks)}$$

S has a singularity when $\cos x = 5/4$, or when $\cosh(ix) = 5/4$ so that $ix = \cosh^{-1}(5/4) = \log(5/4 + \sqrt{(5/4)^2 - 1}) = \log 2$, so that $x = -i\log 2$. (We can also add $2k\pi$ to this value, and as cos is an even function we can multiply x by -1 too. Thus, the nearest singularities to the origin in the complex plane are at $x = \pm i\log 2$. So we expect the Radius of Convergence to be $\log 2$. **(3 marks)**
Alternatively, and more easily, our expansion required $|e^{ix}| < 2$ This breaks down when $ix = \pm\log 2 + 2k\pi i$.