Name (IN CAPITAL LETTERS!): ....

CID: ....

Question 1. When the Experimental Pancake Sampling Research Council visited the department, Ms M. and Mr T. demonstrated their cooking skills. They each folded a nearly circular pancake in two, aligning the diameter with the x-axis, forming a shape given by $r = f(\theta)$ for $0 \leqslant \theta \leqslant \pi$ in terms of usual polar coordinates.

Show that the area of the folded pancake, which is by definition, $A = \int y\, dx$, can be expressed

$$A = \int y\, dx = \int_{\pi}^{0} \sin\theta f(\theta) \Big[\cos\theta f' - \sin\theta f \Big]\, d\theta.$$

Using a suitable integration by parts, deduce from this expression that

$$A = \tfrac{1}{2} \int_{0}^{\pi} f^2\, d\theta.$$

Unfortunately, during the demonstration, as Mr T squashed his pancake in two, a jet of liquid batter erupted, so that his shape was the not very circular

$$f = \sqrt{|\tan\theta|}.$$

Discuss whether or not the integral A exists in this case.

Answer. We have $x = f(\theta)\cos\theta$ and $y = f\sin\theta$. The direction of increasing x runs from $\theta = \pi$ to $\theta = 0$. Thus

$$A = \int_{\pi}^{0} y \frac{dx}{d\theta} d\theta = \int_{\pi}^{0} f \sin\theta (f' \cos\theta - f \sin\theta)\, d\theta \qquad \textbf{(2 marks)}$$

as required. Now noting that ff' is the derivative of $\frac{1}{2} f^2$, integrating the first term by parts, we have

$$A = \left[\sin\theta\cos\theta \tfrac{1}{2} f^2\right]_{\pi}^{0} - \int_{\pi}^{0} \left[\tfrac{1}{2} f^2 (\sin\theta\cos\theta)' + f^2 \sin^2\theta\right] d\theta$$

The first term is zero and using $\cos^2 + \sin^2 = 1$,

$$A = -\int_{\pi}^{0} \tfrac{1}{2} f^2 (\cos^2\theta - \sin^2\theta + 2\sin^2\theta) d\theta = \tfrac{1}{2} \int_{0}^{\pi} f^2\, d\theta. \qquad \textbf{(5 marks)}$$

Now $|\tan\theta|$ is continuous apart from at $\theta = \frac{1}{2}\pi$. Writing $\theta = \frac{1}{2}\pi + t$, $f^2 = |\tan\theta| = |\cos(t)/(-\sin(t))| \sim 1/|t|$ near $t = 0$. This would integrate to $\log|t|$ which is infinite at $t = 0$, so the integral diverges. [They may quote that an algebraic singularity requires a power greater than -1 to be integrable, but they should identify the $1/t$ behaviour.] Thus $A = \frac{1}{2}\int_0^\pi f^2\, d\theta$ diverges, and the integral does not exist. **(3 marks)**

It looks as though Mr. T squandered the entire departmental budget on this single pancake.