Solutions and mark scheme for M1M1 Progress Test 1:

1. (a) Let $y = \exp(x^2 - x)$. Then

$$x^2 - x = \log y \qquad \Longrightarrow x = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + \log y}$$

We must choose the negative sign, given the domain of $f(x)$, so the inverse function is

$$f^{-1}(x) = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + \log x} \qquad \textbf{[5]}$$

We note that this is defined only if both $x > 0$ and $\frac{1}{4} + \log x \geqslant 0$ and so we require $x \geqslant e^{-1/4}$ for f^{-1} to exist. It takes all values in $-\infty < f^{-1}(x) \leqslant \frac{1}{2}$.

(b) Completing the square, we have

$$f(x) = e^{x^2 - x} = e^{(x-1/2)^2 - 1/4} = e^{-1/4} e^{(x-1/2)^2}.$$

Thus $x^2 - x$ is a decreasing function from $x = -\infty$ to $x = 1/2$, where it takes a minimum value, and becomes an increasing function. As e^z is an increasing function of z, it follows that $f(x)$ is decreasing for $-\infty < x \leqslant 1/2$, and an inverse function can be found provided $a \leqslant 1/2$. The maximum value of a is thus $a = \frac{1}{2}$. **[2]**

Then $f(x)$ has domain $x \leqslant \frac{1}{2}$ and range $f(x) \geqslant e^{-1/4}$, whereas $f^{-1}(x)$ has domain $x \geqslant e^{-1/4}$ and range $f^{-1}(x) \leqslant \frac{1}{2}$. **[3]**

2. (a) Expanding the known power series we have

$$g(x) = e^x (\cos x)(1+x)^{-1} = \big[1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+O(x^4)\big]\big[1-\tfrac{1}{2}x^2+O(x^4)\big]\big[1-x+x^2-x^3+O(x^4)\big]$$

[4]

Keeping terms up to and including x^3, we have

$$e^x \cos x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2(1+x) = 1 + x - \tfrac{1}{3}x^3 + O(x^4)$$

Thus

$$\begin{aligned} g(x) &= (1 + x - \tfrac{1}{3}x^3)(1 - x + x^2 - x^3) = 1 - x + x^2 - x^3 + (x - x^2 + x^3) - \tfrac{1}{3}x^3 + O(x^4) \\ &= 1 - \tfrac{1}{3}x^3 + O(x^4) \end{aligned} \qquad \textbf{[6]}$$

3. (a) When $x < 0$, $|x| = -x$ and so $h(x) = \sin 0 = 0$. When $x > 0$, $|x| = x$ and so $h(x) = \sin 2x$. Combining these results gives the graph below. **[3]**

(b) We could write $h_e = \frac{1}{2}(h(x) + h(-x))$ and simplify, but easier is to expand

$$\sin(x + |x|) = \sin x \cos|x| + \cos x \sin|x| = \sin x \cos x + \cos x \sin|x|$$

The first part of this expression is clearly odd and the latter is clearly even.
So $h_e(x) = \cos x \sin|x|$ and $h_o(x) = \sin x \cos x = \frac{1}{2}\sin 2x$. **[4]**

(c) From the graph clearly $h(x)$ is not periodic. $h_o(x)$ is periodic with period π. $h_e(x)$ is not periodic as it behaves strangely near $x = 0$, as shown in lectures. **[3]**

4. The argument is nearly correct. However, when processing equation (2), it is claimed that all the brackets are negative if $56 > x^2 > 6 - \sqrt{12}$ which requires modification. Consider when the first term, a quadratic in x^2, is negative:

$$0 > 24 - 12x^2 + x^4 \iff (x^2 - 6)^2 < 12 \iff -\sqrt{12} < x^2 - 6 < \sqrt{12}$$

Thus not only must we have $x^2 \leqslant 56$, but also $6 - \sqrt{12} \leqslant x^2 \leqslant 6 + \sqrt{12}$. **[5]**
Combining these two constraints gives $6 - \sqrt{12} \leqslant x^2 \leqslant 6 + \sqrt{12}$.

So what is this theorem? It proves $\cos x > 0$ for $0 < x < \sqrt{2} \simeq 1.4$ and $\cos x < 0$ for $6 + \sqrt{12} > x > 1.6$. It follows by continuity that $\cos x$ must be zero somewhere in the range $1.4 < x < 1.6$. Indeed, we know that $\cos \frac{1}{2}\pi = 0$ and $\pi/2 \simeq 1.57$, consistently. **[5]**
We could use this as a definition of π: Twice the first zero of $\cos x$ (defined by its power series).

Total : 40

General instructions to markers

Each script should be awarded an integer mark between 0 & 20 inclusive. Take the total out of 40 and divide by 2, rounding up or down using your judgement.

You should deduct marks for illegibly named scripts and may penalise mathematical incoherence. Well presented arguments may deserve credit even if marred by algebraic or arithmetical slips. You may award a bonus mark for ‘good’ mathematics. Correct, but unexplained answers may not deserve full credit, at your discretion. Not everything in the model answers is needed for full credit.

Question 4 in particular may require you to use your judgement. Try to give helpful comments to the students; they will receive a copy of this sheet eventually.