M1M1 Progress Test 1: Friday, October 26th 2007.

Write your name **clearly** on your answer book.

No calculators, books or notes. 50 minutes. Attempt all four questions.

1. The function $f(x)$ is defined as

$$f(x) = \exp(x^2 - x) \qquad \text{for} \quad -\infty < x < a,$$

where a is some positive number.

(a) Find the inverse function, $f^{-1}(x)$, assuming it exists.

(b) What is the largest possible value of a for which f^{-1} exists? Assuming a takes this value, what are the domain and range of $f(x)$ and $f^{-1}(x)$?

2. The function $g(x)$ is defined by

$$g(x) = \frac{e^x \cos x}{1 + x}$$

Find the power series expansion of $g(x)$, neglecting terms of $O(x^4)$.

3. Consider the function defined for all x

$$h(x) = \sin(x + |x|)$$

(a) Sketch the curve $y = h(x)$ over the domain $-2\pi < x < 2\pi$.

(b) Find an even function $h_e(x)$ and an odd function $h_o(x)$ such that $h(x) = h_e(x) + h_o(x)$, simplifying your answers where possible.

(c) Which (if any) of $h(x)$, $h_e(x)$ and $h_o(x)$ is periodic?

4. **Turn over for question 4.**

4. The following argument contains one slight mistake somewhere after equation (2). It is also missing the last line. Correct the mistake and provide the conclusion. Compare the conclusion with what you know about $\cos x$.

A little theorem with a mistake

The power series for the function $\cos x$ can be rearranged in the two equivalent forms:

$$\cos x = \left[1 - \frac{x^2}{2}\right] + \frac{x^4}{4!}\left(1 - \frac{x^2}{(5)(6)}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{(9)(10)}\right) + \dots \quad (1)$$

or alternatively

$$\cos x = \frac{1}{4!}\left[24 - 12x^2 + x^4\right] - \frac{x^6}{6!}\left(1 - \frac{x^2}{(7)(8)}\right) - \frac{x^{10}}{10!}\left(1 - \frac{x^2}{(11)(12)}\right) + \dots \quad (2)$$

Now from equation (1), we can show that $\cos x > 0$ if

$$0 < x < \sqrt{2} \simeq 1.414,$$

since then all the bracketed terms are positive.

Similarly, analysing the first term in equation (2), we can show that $\cos x < 0$ provided

$$56 > x^2 > (6 - \sqrt{12}),$$

since then all the terms are negative.

We note that $(6 - \sqrt{12})^{1/2} \simeq 1.592$.

Conclusion: Since $\cos x$ is a continuous function, ...

QE(almost)D