Solutions and mark scheme for M1M1 Progress Test 1:

1. (a) e^{-x} is a decreasing function taking all positive values as x varies. Therefore

$$f(x) = \frac{1}{\sqrt{1+e^{-x}}} \qquad \text{satisfies} \qquad 0 < f(x) < 1.$$

So the range of $f(x)$ is $0 < f < 1$. **[2]**
(Furthermore $f(x)$ is an increasing function and so its inverse exists over the entire range.)

(b) Writing $y = f(x)$ and squaring (we must then subsequently remember that $y > 0$)

$$y^2 = \frac{1}{1+e^{-x}} \qquad \Longrightarrow \qquad e^{-x} = \frac{1}{y^2} - 1 = \frac{1-y^2}{y^2}$$

Thus

$$e^x = \frac{y^2}{1-y^2} \qquad \Longrightarrow \qquad x = \log\left[\frac{y^2}{1-y^2}\right] \equiv f^{-1}(y).$$

Therefore

$$f^{-1}(x) = \log\left[\frac{x^2}{1-x^2}\right].$$ **[2]**

(c) Now the function $\log(u)$ is defined only for $u > 0$. So the domain of $f^{-1}(x)$ requires $x^2/(1-x^2) > 0$ or $|x| < 1$ (and $x \neq 0$). However, since the range of $f(x)$ is $0 < f(x) < 1$, we know $f^{-1}(x)$ is only defined for $0 < x < 1$ and so this is the domain of f^{-1}. **[1]**

The range of $f^{-1}(x)$ is the set of values it attains for all x in its domain. Recalling that "$\log(0) = -\infty$", this is $-\infty < f^{-1}(x) < +\infty$ (the same as the domain of $f(x)$.) **[1]**

2. (a) We know x and $\sin x$ are odd functions, and x^2 is even. Thus $x \sin x/(1-x^2)$ is an even function, while $x^2 \sin x/(1-x^2)$ is odd. It follows that

$$g_e(x) = \frac{x \sin x}{1-x^2} \qquad g_o(x) = \frac{x^2 \sin x}{1-x^2} \qquad g(x) = g_e(x) + g_o(x).$$ **[2]**

(b) Now

$$\begin{aligned}\frac{x \sin x}{1-x^2} &= x\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots\right)(1 + x^2 + x^4 + \ldots) \\ &= x^2 - \frac{x^4}{6} + \frac{x^6}{120} + x^4 - \frac{x^6}{6} + x^6 + O(x^8) \\ &= x^2 + \tfrac{5}{6}x^4 + \tfrac{101}{120}x^6 + O(x^8)\end{aligned}$$ **[2]**

(c) As $g_o(x) = x g_e(x)$ it follows that

$$g_o(x) = x^3 + \tfrac{5}{6}x^5 + \tfrac{101}{120}x^7 + O(x^9).$$ **[1]**

3. (a) We know that

$$f(y) \equiv \frac{1}{1-y} = 1 + y + y^2 + \ldots = \sum_{n=0}^{\infty} y^n,$$

and so

$$\begin{aligned} f(x+2x^2) &= 1 + (x+2x^2) + (x+2x^2)^2 + (x+2x^2)^3 + \dots \\ &= 1 + x + 3x^2 + 5x^3 + O(x^4) \end{aligned} \qquad [\mathbf{2}]$$

Now

$$\begin{aligned} f(x+2x^2) &= \frac{1}{1-x-2x^2} = \frac{1}{(1+x)(1-2x)} = \frac{1/3}{1+x} + \frac{2/3}{1-2x} \\ &= \tfrac{1}{3}\sum_{n=0}^{\infty}(-x)^n + \tfrac{2}{3}\sum_{n=0}^{\infty}(2x)^n = \tfrac{1}{3}\sum_{n=0}^{\infty}\left(2^{n+1} + (-1)^n\right)x^n \ , \end{aligned} \qquad [\mathbf{3}]$$

as required.

(c) When $n = 0$, 1, 2, 3, we find $2^{n+1} + (-1)^n = 3, 3, 9, 15$ so (a) and (b) agree. [**1**]

4. As presented, the argument misuses the $\sum$ notation dreadfully. In line 4 $(m+n)!$ is taken outside the summation over n. Then the binomial theorem is used with n running from 0 to $m+n$, even though n never reaches $m+n$ for $m > 0$. Finally, the definition of exp is used as if the summation was $\sum_{m+n=0}^{\infty}$ rather than $\sum_{m=0}^{\infty}$. [**3**]

(Nevertheless, the proof is mendable. What is missing is a step arguing that the summation over all integers m and n is equivalent to a summation over all possible totals $t = m + n$ and then all possible ways of reaching that total. In other words, it must be argued that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\dots = \sum_{t=0}^{\infty}\sum_{n=0}^{t}\dots$$

Then the proof would be correct. This is equivalent to grouping together all the terms of the same degree, as on the solutions to problem sheet 1. [**Bonus 2**])

Total : 20 + 2

General instructions to markers

Each script should be awarded an integer mark between 0 & 20 inclusive. You may use fractional marks in the middle if you wish, and then round to an integer using your judgement.

You should deduct marks for illegibly named scripts and may penalise mathematical incoherence. Well presented arguments may deserve credit even if marred by algebraic or arithmetical slips. You may award a bonus mark for 'good' mathematics. Correct, but unexplained answers may not deserve full credit, at your discretion.

Question 4 in particular will require you to use your judgement. Try to give helpful comments to the students; they will receive a copy of this sheet eventually.