M1M1 Progress Test 1: October 27th 2006.

Write your name **clearly** on your answer book.

No calculators, books or lecture notes. 50 minutes. Attempt all four questions.

1. The function $f(x)$ is defined as

$$f(x) = \frac{1}{\sqrt{1+e^{-x}}}, \qquad \text{for all } x.$$

(a) What is the range of $f(x)$?
(b) Find the inverse function $f^{-1}(x)$.
(c) What are the range and domain of $f^{-1}(x)$?

2. The function $g(x)$ is defined by

$$g(x) = \frac{(x+x^2)\sin x}{1-x^2}$$

(a) Express $g(x)$ in the form $g(x) = g_e(x) + g_o(x)$ where $g_e(x)$ is even and $g_o(x)$ is odd.
(b) Find the first 3 non-zero terms in the power series for $g_e(x)$.
(c) Deduce the first 3 non zero terms in the series for $g_o(x)$,

3. Write down the power series in y for

$$f(y) = \frac{1}{1-y}.$$

(a) By substituting $y = x + 2x^2$ in this series, obtain the first four non-zero terms in the series for

$$g(x) \equiv f(x+2x^2).$$

(b) Rewrite $g(x)$ in partial fraction form, and hence show that

$$g(x) = \tfrac{1}{3}\sum_{n=0}^{\infty}\left[2^{n+1} + (-1)^n\right]x^n.$$

(c) Verify that your answers for (a) and (b) are consistent.

4. **Turn over for question 4.**

4. The following argument is proposed for proving that $\exp(x)\exp(y) = \exp(x+y)$ where the function exp is defined from a power series.

Explain in a few sentences what is wrong with the "proof" as it stands.
Bonus marks may be awarded for "mending" the proof.

An attempt to prove that $\exp(x)\exp(y) = \exp(x+y)$

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\begin{aligned}
\exp(x)\exp(y) &= \left[\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)\right]\left[\sum_{m=0}^{\infty}\left(\frac{y^m}{m!}\right)\right] && \text{(by the power series definition)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!\,n!}\,\frac{x^n y^m}{(m+n)!} && \text{(rearranging terms)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{x^n y^m}{(m+n)!} && \text{(by the binomial coefficient definition)}\\
&= \sum_{m=0}^{\infty}\frac{1}{(m+n)!}\sum_{n=0}^{\infty}\binom{m+n}{n}x^n y^{(m+n)-n} && \text{(rearranging terms)}\\
&= \sum_{m=0}^{\infty}\frac{1}{(m+n)!}(x+y)^{m+n} && \text{(by the Binomial theorem)}\\
&= \exp(x+y) && \text{(from the definition of the function exp.)}
\end{aligned}
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QE(almost)D