M1M1 Handout: Ordinary Differential Equations, or ODEs

An ODE is an equation relating an unknown function, say $y(x)$, and at least one of its derivatives $(y'(x), y''(x)\dots)$ The **order** of the ODE is the number of the highest derivative to occur in the equation. x is usually called the **independent variable**, y the **dependent variable.**

A collection of all possible solutions to an ODE is called its **general solution.** The general solution of an n^{th} order ODE has n arbitrary constants in it, one for each integration which occurs during the solution. To determine the solution uniquely n extra pieces of information are required, so that for a first order equation we may need the value of y at one specific point.

We regard an ODE as 'solved' if the derivatives can somehow be removed leaving an equation relating x and y alone, possibly involving integrals of **known** functions. We shall not worry about whether the integrals involved can be evaluated in terms of simple functions.

Linear ODEs

We say an ODE is **linear** if y and its derivatives appear as a linear combination, with coefficients which may depend on x. Thus the most general n^{th} order ODE can be written

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x) \qquad (1)$$

for known functions a_0, $a_1, \dots a_n$ and f. If $f \equiv 0$, we say the equation is **homogeneous**, whereas if $f \neq 0$ we say it is **inhomogeneous**. If $n > 1$, this equation cannot be solved in general, but we can say a lot about the structure of the solution.

Suppose we can find n **linearly independent** solutions of (1) when $f \equiv 0$, which we write as $y = \phi_i(x)$ for known functions ϕ_1, $\phi_2 \dots$ for $i = 1 \dots n$. Suppose further that we can find one particular solution of the full equation (1), which we write as $y = P(x)$. Then the general solution of (1) can be written as

$$y = A_1\phi_1(x) + A_2\phi_2(x) + \dots A_n\phi_n(x) + P(x), \qquad (2)$$

where A_1, $A_2 \dots$ are arbitrary constants. In this expression P is often called the **Particular Integral** (or P.I.), and the rest of the solution is called the **Complementary Function** (or C.F.). We shall not discuss Particular Integrals in this course – see M1M2 for further details. [Note: In the language of M1GLA, The Complementary Function is a **Vector Space** with basis $\{\phi_1, \phi_2, \dots \phi_n\}$.]

Linear ODEs with Constant Coefficients

In general, the functions ϕ_i are not easy to find. However, if the functions a_i are all constants, then there is a simple method of finding them.

A little thought indicates that if $y = e^{px}$, where p is a constant, then $y^{(n)} = p^n y$. It follows that $y = e^{px}$ is a solution of (1) with $f = 0$ provided

$$\left[a_n p^n + a_{n-1}p^{n-1} + \dots + a_1 p + a_0\right] e^{px} = 0. \qquad (3)$$

This holds if p is one of the n roots of the polynomial in square brackets. In general, there are n roots of this polynomial, and we have our n linearly independent solutions ϕ_n. The polynomial is known as the **auxiliary equation.**

Example: let $y'' + y' - 2y = 0$. Try $y = \exp(\lambda x)$. We obtain the auxiliary equation $\lambda^2 + \lambda - 2 = 0$, which has roots $\lambda = 1$, and $\lambda = -2$. The general solution is then

$$y = Ae^x + Be^{-2x}.$$

Complex Roots:

The algebra goes through if the auxiliary equation has complex roots, which indicate trigonometric functions. For example, with $y = \exp(rx)$.

$$y'' + y = 0 \quad \Longrightarrow \quad r^2 + 1 = 0 \quad \Longrightarrow \quad r = \pm i.$$

Thus

$$y = Ae^{ix} + Be^{-ix} = C\cos x + D\sin x,$$

where C and D are arbitrary, related to the arbitrary A and B by $C = A + B$ and $D = i(A - B)$. If instead, the two roots are fully complex, say, $a \pm ib$, then the solution would be $\exp(ax)(C\cos bx + D\sin bx)$.

Repeated Roots:

If the auxiliary equation has a repeated root, λ_0 say, then two of the ϕ_i would be the same. In this case, a simple practical rule is to try multiplying the repeated exponential by x. Question 2 on Problem Sheet 9 justifies this rule. Thus for example, the third order ODE

$$y''' + 3y'' + 3y' + y = 0 \qquad \text{has the auxiliary equation} \quad (p+1)^3 = 0$$

which has a triple root. Its general solution is

$$y = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x} = e^{-x}(A + Bx + Cx^2),$$

as can be verified by substitution.

Once the General Solution has been obtained, a particular solution which satisfies extra conditions may be sought. For example, suppose

$$y'' + 4y' + 13y = 0 \qquad \text{with} \quad y(0) = 3,\ y'(0) = 0.$$

The General Solution is $y = e^{-2x}(A\cos 3x + B\sin 3x)$. Imposing the **boundary condition** (or **initital condition**) $y(0) = 3$ gives $3 = A$. Differentiating, we have

$$y' = e^{-2x}\left[-2A\cos 3x - 2B\sin 3x - 3A\sin 3x + 3B\cos 3x\right] \quad \Longrightarrow \quad y'(0) = -2A + 3B.$$

Imposing the second boundary condition, we have $0 = -2(3) + 3B$ or $B = 2$. So the solution is

$$y = e^{-2x}\left[3\cos 3x + 2\sin 3x\right].$$

First Order ODEs

We now turn our attention to 1st order ODEs, i.e. equations relating y, x and y'. These can often be solved exactly. There are several easily recognisable types of 1st order ODEs which can be solved systematically. We examine these in turn.

(A) Separable equations:

An equation which can be written in the form

$$\frac{dy}{dx} = f(x)g(y) \qquad \text{is called } \textbf{separable}. \tag{4}$$

It is easily solved by writing the x and y parts on opposite sides of the equation and integrating,

$$\int f(x)dx = \int \frac{dy}{dx}\frac{dx}{g(y)} = \int \frac{dy}{g(y)}. \tag{5}$$

(B) Linear First Order ODEs:

An equation is called linear if it can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x). \tag{6}$$

Such equations can always be solved by multiplying them by a suitable **integrating factor,** $I(x)$, so that

$$Iy' + pIy = qI. \qquad \text{We try to choose } I(x) \text{ so that the LHS is } (Iy)'. \tag{7}$$

Expanding $(Iy)'$, we see that this requires

$$I'y = pIy \qquad \Longrightarrow \qquad \frac{1}{I}\frac{dI}{dx} = p(x). \tag{8}$$

which is a separable equation for the function $I(x)$. Integrating, we have

$$\int \frac{dI}{I} = \int p(x)\,dx \qquad \Longrightarrow \qquad I(x) = \exp\left[\int p(x)\,dx\right]. \tag{9}$$

Having found $I(x)$, the equation (8) takes the form

$$\frac{d}{dx}\left(I(x)y\right) = q(x)I(x) \qquad \Longrightarrow \qquad y = \frac{1}{I(x)}\int qI\,dx + c. \tag{10}$$

As ever, it is more important to remember the idea behind the method, than the formula itself. The most common error when finding integrating factors is to forget to divide through by a function multiplying y'. Note that the integrating factor may be multiplied by an arbitrary constant, which derives from the integration in (9).

(C) Dimensionally Homogeneous First Order ODEs:

It is very unfortunate that standard ODE terminology uses the word "Homogeneous" to mean two different things. The first meaning (see equation (1)) is the more general one. Any problem for which the zero function is a permissible solution can be called homogeneous. We shall distinguish the two cases by calling this case **dimensionally homogeneous.**

An equation of the form for some function f

$$\frac{dy}{dx} = f(y/x) \qquad \text{is called } \textbf{dimensionally homogeneous.} \tag{11}$$

If you imagine x and y have physical dimensions, say length, this means that every term in the equation has the same physical dimension. Equations of this type can always be solved by the substitution

$$y(x) = xv(x) \qquad \Longrightarrow \qquad y' = xv' + v. \tag{12}$$

Equation (11) then takes the separable form

$$x\frac{dv}{dx} + v = f(v) \qquad \Longrightarrow \qquad \int \frac{dv}{f(v) - v} = \int \frac{dx}{x}. \tag{13}$$

Having integrated to find v, the general solution for y follows.

(D) Substitutions: Various tricks

Sometimes equations can be transformed into recognisable forms by a suitable substitution. There are no rules here – use your imagination and judgement. Here are a few examples:

$$y\frac{dy}{dx} + xy^2 = x^2 \qquad \text{Try } v = y^2 \tag{14}$$

$$\frac{dy}{dx} = a(x)y + b(x)y^2 \qquad \text{Try } v = y^{-1} \tag{15}$$

$$\frac{dy}{dx} = \frac{1}{x + e^y} \qquad \text{Try solving for } x \text{ as a function of } y \tag{16}$$

These last three examples can be transformed to linear equations in the new variables.

$$\frac{dy}{dx} = \frac{x + 2y + 3}{3x + y + 4} \quad \text{Try } X = x + a,\ Y = y + b \text{ to get a homogeneous equation in } Y(X) \tag{17}$$

A trick worth noting is that d^2y/dx^2 can be written as $p\,dp/dy$ where $p = dy/dx$. So for example, if x is some function of dy/dx, the solution can be found parametrically in terms of p:

$$x = f(p) \qquad \Longrightarrow \qquad 1 = f'(p)\frac{dp}{dx} = f'(p)p\frac{dp}{dy} \qquad \Longrightarrow \qquad y = \int f'(p)p\,dp. \tag{18}$$