1. (a) Using any method, find the derivative of the inverse hyperbolic tangent,

$$\frac{d}{dx}\left[\tanh^{-1}(x)\right].$$

(b) A maths lecturer falls from an aeroplane under gravity with air resistance. His/her speed V varies with time according to the equations

$$\frac{dV}{dt} + kV^2 = g, \qquad V(0) = 0,$$

where g and k are positive constants. Find $V(t)$. [You may find part (a) useful.]

(c) If $V = dx/dt$ and $x(0) = 0$, show that

$$x(t) = \frac{1}{k}\log\left[\cosh\left(t\sqrt{kg}\right)\right]$$

(d) Find the limit of $x(t)$ as $k \to 0$.

(e) As $t \to \infty$ in part (c) with k fixed, find the constants A and B such that $x \simeq At + B$.

(f) Find all complex numbers z such that $\cosh z = 0$.

(g) If we write $x(t)$ as a power series $\sum a_n t^n$, what do you expect the Radius of Convergence of this series, R, to be?

2\. (a) Suppose f and g are continuous functions on an interval $[a, b]$, and λ is an arbitrary parameter. By writing the (positive) integral

$$\int_a^b [f(x) + \lambda g(x)]^2 \, dx \qquad \text{as a quadratic in } \lambda,$$

prove that

$$\left[\int_a^b fg \, dx\right]^2 \leqslant \left(\int_a^b f^2 \, dx\right)\left(\int_a^b g^2 \, dx\right).$$

Deduce that

$$\int_0^1 \frac{e^x}{x+1} \leqslant \tfrac{1}{2}\sqrt{e^2 - 1}.$$

(b) If $y = \sin^{-1} x + (\sin^{-1} x)^2$, show that $(1 - x^2)y'' - xy'$ is a constant. Hence find a relation between the n'th, $(n + 1)$'th and $(n + 2)$'th derivative, and give an expression for $y^{(n)}(0)$ if n is odd.

(c) Write the function $y(x)$ from part (b) as the sum of an even part, $y_e(x)$ and an odd part $y_o(x)$. Sketch y_e and y_o between $x = \pm 1$ on the same diagram, identifying any turning points, intersections and behaviour at singularities.

Solutions

1. (a) If $y = \tanh^{-1} x$ then $x = \tanh y$ and $dx/dy = \operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$. Thus

$$\frac{d}{dx}\left[\tanh^{-1}(x)\right] = \frac{1}{1-x^2}.$$ **[2 marks]**

(b) Separating, and substituting $V = (g/k)^{1/2} u$ we have

$$\int \frac{dV}{g - kV^2} = \int dt \qquad \Longrightarrow \qquad t = \left(\frac{g}{k}\right)^{1/2} \int \frac{du}{g(1-u^2)} = \frac{1}{\sqrt{gk}} \tanh^{-1} u + C$$

Now when $t = 0$, $V = 0 = u$ so that $C = 0$ and we have **[1 mark]**

$$V = \left(\frac{g}{k}\right)^{1/2} \tanh\left(\sqrt{kg}t\right).$$ **[3 marks]**

[Anyone who uses a log rather than $\tanh^{-1}$ may still earn full marks, provided they simplify a reasonable amount.]

(c) Integrating again, we have

$$x = \left(\frac{g}{k}\right)^{1/2} \frac{1}{(gk)^{1/2}} \log[\cosh\left(\sqrt{kg}t\right) + A$$

and since $x = 0$ when $t = 0$, we have $A = 0$. **[1 mark]**
Thus

$$x(t) = \frac{1}{k} \log\left[\cosh\left(t\sqrt{kg}\right)\right].$$ **[3 marks]**

(d) As $z \to 0$, we have $\cosh z = 1 + \frac{1}{2}z^2 + O(z^4)$ and so $\log(\cosh z) = \frac{1}{2}z^2 + O(z^4)$. Therefore

$$x = \frac{1}{k}[\tfrac{1}{2}kgt^2 + O(k^2g^2t^4)] = \tfrac{1}{2}gt^2$$ **[3 marks]**.

[As expected if we did A-level mechanics...]

(e) As $z \to \infty$, $\cosh z \simeq \frac{1}{2}e^z$ and so $\log(\cosh z) \simeq z - \log 2$. Thus

$$x \simeq \frac{1}{k}(t\sqrt{kg} - \log 2) = t\sqrt{\frac{g}{k}} - \frac{1}{k}\log 2.$$ **[3 marks]**

(f) If $\cosh z = 0$ then $e^z + e^{-z} = 0$ or $e^{2z} = -1 = e^{i\pi}$. Taking logs, we have for integer n,

$$2z = i\pi + 2n\pi i \qquad \Longrightarrow \qquad z = i\pi(n + \tfrac{1}{2})$$ **[2 marks]**.

(g) We expect the power series to converge in as large a circle as it can until it hits a singularity in the complex plane. Now $\log(\cosh z)$ is infinite when $\cosh z = 0$. The closest singularity to the origin is at $z = \pm\frac{1}{2}\pi i$. Thus we expect the series for $x(t)$ to converge for

$$|\sqrt{gk}t| < \tfrac{1}{2}\pi \qquad \Longrightarrow \qquad |t| < \frac{\pi}{2\sqrt{gk}} = R.$$ **[2 marks]**

2. (a) The integral in question is positive or zero for every λ. Further, it can be written as a quadratic $P(\lambda) \equiv A\lambda^2 + B\lambda + C$ where

$$A = \int g^2\,dx, \qquad B = 2\int fg\,dx, \qquad C = \int f^2\,dx.$$

Now since $P(\lambda) \geqslant 0$ for all λ, we must have $A \geqslant 0$ and $B^2 \leqslant 4AC$. The first is clearly true as A is the integral of a square, while the second requires

$$\left[\int_a^b fg\,dx\right]^2 \leqslant \left(\int_a^b f^2\,dx\right)\left(\int_a^b g^2\,dx\right).$$ **[4 marks]**

Now let $f(x) = e^x$ and $g(x) = 1/(x+1)$. Then $A = \frac{1}{2}(e^2 - 1)$ while $C = [-(x+1)^{-1}]_0^1 = \frac{1}{2}$. Thus

$$\left(\int_0^1 \frac{e^x}{x+1}\,dx\right)^2 \leqslant \tfrac{1}{4}(e^2 - 1),$$ **[2 marks]**

and the result follows.

(b) We have

$$y' = (1 + 2\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} \qquad \Longrightarrow \qquad \sqrt{1-x^2}\,y' = 1 + 2\sin^{-1} x.$$

Differentiating again,

$$\sqrt{1-x^2}\,y'' - \frac{xy'}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} \qquad \Longrightarrow \qquad (1-x^2)y'' - xy' = 2.$$ **[3 marks]**

Differentiating n times by Leibniz, we have

$$(1-x^2)y^{(n+2)} - 2xny^{(n+1)} - 2n(n-1)/2y^{(n)} - xy^{(n+1)} - ny^{(n)} = 0$$

or

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0.$$ **[3 marks]**

Putting $x = 0$, we have

$$y^{(n+2)}(0) = n^2y^{(n)}(0).$$

Now $y'(0) = 1$. Thus if n is odd,

$$y^{(n)}(0) = (n-2)^2(n-4)^2\ldots3^2y'(0) = (n-2)^2(n-4)^2\ldots3^2.$$ **[2 marks]**

[This can also be written in terms of factorials. If n is even (not asked) a similar result holds - no extra credit.]]

(c) Since $\sin^{-1} x$ is odd, we see by inspection that $y_e = (\sin^{-1} x)^2$ and $y_o = \sin^{-1} x$. $y = y_o(x)$ is the reflection of the curve $y = \sin x$ in the line $y = x$. It passes through the origin and $(1, \frac{1}{2}\pi)$. It is an increasing function. Near the origin, $y_o \simeq x$.

Similarly, y_e is increasing from $(0, 0)$ to $(1, \frac{1}{4}\pi^2)$. Near the origin $y_e \simeq x^2$. It follows that the two curves intersect at $(\sin 1, 1)$. At $x = 1$, both have infinite gradient.

The behaviour for $x < 0$ follows from the parity. **[6 marks]**