Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2013

M1M1 (January Test)

Mathematical Methods I

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Calculators may not be used.

1. *You may assume standard properties of e^x in this question, and you need not check too carefully whether an appropriate stationary value is a maximum or minimum.*

(a) Some non-mathematicians find the concept of variables confusing, and prefer to treat all functions as constants. Thus, for example, they approximate e^x over the domain $(0,\, 1)$ by a constant, A. One way to choose the optimal A is to minimise the "root-mean-square error," E_1, where

$$(E_1)^2 = \int_0^1 (e^x - A)^2 \; dx.$$

Find the minimising value of A and the corresponding minimum error E_1.

(b) Another way to define the "best" constant approximation is to approximate the curve $y = e^x$ by $y = B$, where B minimises the modulus error, E_2, given by

$$E_2 = \int_0^1 |e^x - B| \, dx.$$

Find the optimal B and minimum value of E_2 (you may assume that $e^0 < B < e^1$).

(c) A third approximation chooses a point C in $0 < C < 1$. Write down the **linear** function, $f_C(x)$, formed by the first two terms in the Taylor series of e^x about $x = C$. One can then define the error

$$E_3 = \int_0^1 (e^x - f_C(x)) \; dx.$$

Calculate the value of C which minimises E_3 and also the minimum value.

(d) "This is ridiculous!" you cry. "The curve $y = e^x$ is not a straight line. It has curvature, K, given by

$$K = \frac{y''}{[1 + (y')^2]^{3/2}}.\text{ "}$$

What is the maximum value taken by the curvature over the domain $0 \leqslant x \leqslant 1$?

(e) Find all complex numbers x such that both $|x| = 1$ and $|e^x| = 1$.

2. (a) Find the general solution $y(x)$ of the ODE

$$y'' + 2y' + (1 + b^2)y = 0$$

where b is a positive constant.

(b) Find the solution to part (a) which also satisfies the conditions $y(0) = 1$ and $y'(0) = 1$.

(c) Take the limit of the solution in part (b) as $b \to 0$. Verify that the limiting function you obtain satisfies the ODE when $b = 0$.

(d) Sketch the curve $y = (1 + 2x)e^{-x}$ identifying any stationary points and points of inflexion.

Solutions

1. (a) Minimising the positive function $E_1(A)$ is the same as minimising E_1^2. Now

$$E_1^2 = \int_0^1 \left(e^{2x} - 2Ae^x + A^2\right) dx = \tfrac{1}{2}(e^2-1) - 2A(e-1) + A^2 = [A-(e-1)]^2 - \tfrac{1}{2}e^2 + 2e - \tfrac{3}{2}.$$

Thus the minimum occurs when $A = e - 1$ **[3 marks]**
and the minimum value is $E_1 = \sqrt{2e - e^2/2 - 3/2} = \sqrt{(e-1)(3-e)/2}$. **[1 mark]**

(b) Splitting the integral into two ranges where $e^x > B$ and $e^x < B$, we have

$$E_2 = \int_0^{\log B} (B - e^x)\, dx + \int_{\log B}^1 (e^x - B)\, dx = B\log B - (B-1) + (e-B) - B(1 - \log B)$$

$$= -3B + (1+e) + 2B\log B.$$

Differentiating w.r.t B, we have

$$\frac{dE_2}{dB} = 2\log B - 1 = 0 \qquad \text{when} \quad B = \sqrt{e}.$$ **[3 marks]**

At this value, $E_2 = -2\sqrt{e} + 1 + e = (\sqrt{e} - 1)^2$. **[1 mark]**

(c) The Taylor series of $f(x)$ about $x = c$ is $f(x) = f(c) + f'(c)(x-c) + \ldots$ Thus $f_C(x) = e^C[1 + (x - C)]$. **[1 mark]**

$$E_3 = \int_0^1 \left(e^x - e^C(1 - C + x)\right) dx = (e-1) - e^C(1-C) - \tfrac{1}{2}e^C.$$

E_3 is minimum when

$$0 = \frac{dE_3}{dC} = e^C\left[C - \tfrac{1}{2}\right] \qquad \text{or} \quad C = \tfrac{1}{2}.$$ **[3 marks]**

The $E_3 = e - 1 - \sqrt{e}$. **[1 mark]**

(d) We have

$$\frac{dK}{dx} = \frac{e^x}{(1+e^{2x})^{3/2}} - \frac{3e^{3x}}{(1+e^{2x})^{5/2}} = \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}}.$$

Thus the stationary curvature is when $x = -\frac{1}{2}\log 2$ and takes the value

$$K = 1/\sqrt{2}(2/3)^{3/2} = 2\sqrt{3}/9.$$

However, the question is only interested in the interval $[0,1]$, when $dK/dx < 0$. It follows that the maximum value occurs at $x = 0$, and takes the value $1/(2\sqrt{2}) = \frac{1}{4}\sqrt{2}$. **[4 marks]** – for giving instead the correct global maximum award 2 marks.

(e) If $|x| = 1$, then we can write $x = e^{i\theta} = \cos\theta + i\sin\theta$. Then $|e^x| = e^{\cos\theta} = 1$ when $\cos\theta = 0$, which means $\sin\theta = \pm 1$. Thus the only complex numbers with these properties are $x = \pm i$. **[3 marks]**

2. (a) Seeking solutions $y = e^{mx}$, we obtain the auxiliary equation

$$m^2 + 2m + (1 + b^2) = 0 \qquad \Longrightarrow \qquad m = -1 \pm ib.$$

Thus the general solution is

$$y = Ae^{(-1+ib)x} + Be^{(-1-ib)x} = e^{-x}\left[C\cos bx + D\sin bx\right].$$ **[5 marks]**

(b) Imposing $y(0) = 1$, we have $C = 1$. Differentiating, we have

$$y' = e^{-x}\left[-C\cos bx - D\sin bx - bC\sin bx + bD\cos bx\right].$$

Imposing $y'(0) = 1$, we have

$$1 = -C + bD \qquad \Longrightarrow \qquad D = 2/b.$$

Thus we have the particular solution

$$y = e^{-x}\left[\cos bx + \frac{2}{b}\sin bx\right].$$ **[5 marks]**

(c) As $b \to 0$, we have $\sin bx \simeq bx$, and so

$$y \to e^{-x}(1 + 2x).$$ **[3 marks]**

With $y = e^{-x}(1 + 2x)$, we have $y' = e^{-x}(1 - 2x)$ and $y'' = e^{-x}(2x - 3)$. So

$$y'' + 2y' + y = e^{-x}\left[2x - 3 + 2(1 - 2x) + 1 + 2x\right] = 0.$$

So this satisfies the ODE as required. **[1 mark]**

(d) From above, $y' = 0$ when $x = 1/2$ at which point $y'' < 0$ giving a maximum at $(1/2,\, 2e^{-1/2})$. Furthermore, $y'' = 0$ when $x = 3/2$, so we expect a point of inflexion at $(3/2,\, 4e^{-3/2})$. We know that $y(0) = 1 = y'(0)$, $y(-1/2) = 0$ and also that $y \to 0$ as $x \to \infty$, while $y \to -\infty$ very rapidly as $x \to -\infty$. The graph follows. **[6 marks]**