Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2012

# M1M1 (January Test)

# Mathematical Methods I

- Write your name, College ID, Personal Tutor, and the question number prominently on the front of each answer book.
- Write answers to each question in a separate answer book.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

## SECTION A

1. **Introduction:** *A certain lecturer's handwriting is so ba, that it is difficult to tell whether they have written* $\sin(\cos x)$ *or* $\cos(\sin x)$*. Having nothing else to do for the next half hour, you wonder whether there are any values of* $x$ *for which it doesn't matter:*

(a) The four functions $\cos(\cos x)$, $\cos(\sin x)$, $\sin(\cos x)$ and $\sin(\sin x)$, are plotted in the figure. The bottom left corner is at $(0, 0)$.

Determine which curve corresponds to which function, and identify the points where the curves cross each other.

(b) Express $\cos(\sin x)$ as a power series about $x = 0$, giving terms up to and including $x^4$.

(c) If $\cos x = \cos\alpha$ write down all possible real values of $x$ in terms of $\alpha$.
Noting that $\sin t = \cos(\frac{1}{2}\pi - t)$, show that if

$$\cos(\sin x) = \sin(\cos x), \qquad \text{then}$$

$$\cos(x \pm \tfrac{1}{4}\pi) = \frac{(4n+1)\pi}{2\sqrt{2}},$$

where $n$ is an integer. Hence find all real values of $x$ such that $\cos(\sin x) = \sin(\cos x)$, and verify this agrees with the sketch in part (a).

(d) Find all complex numbers $z$ such that if $\beta$ is real with $\beta > 1$,

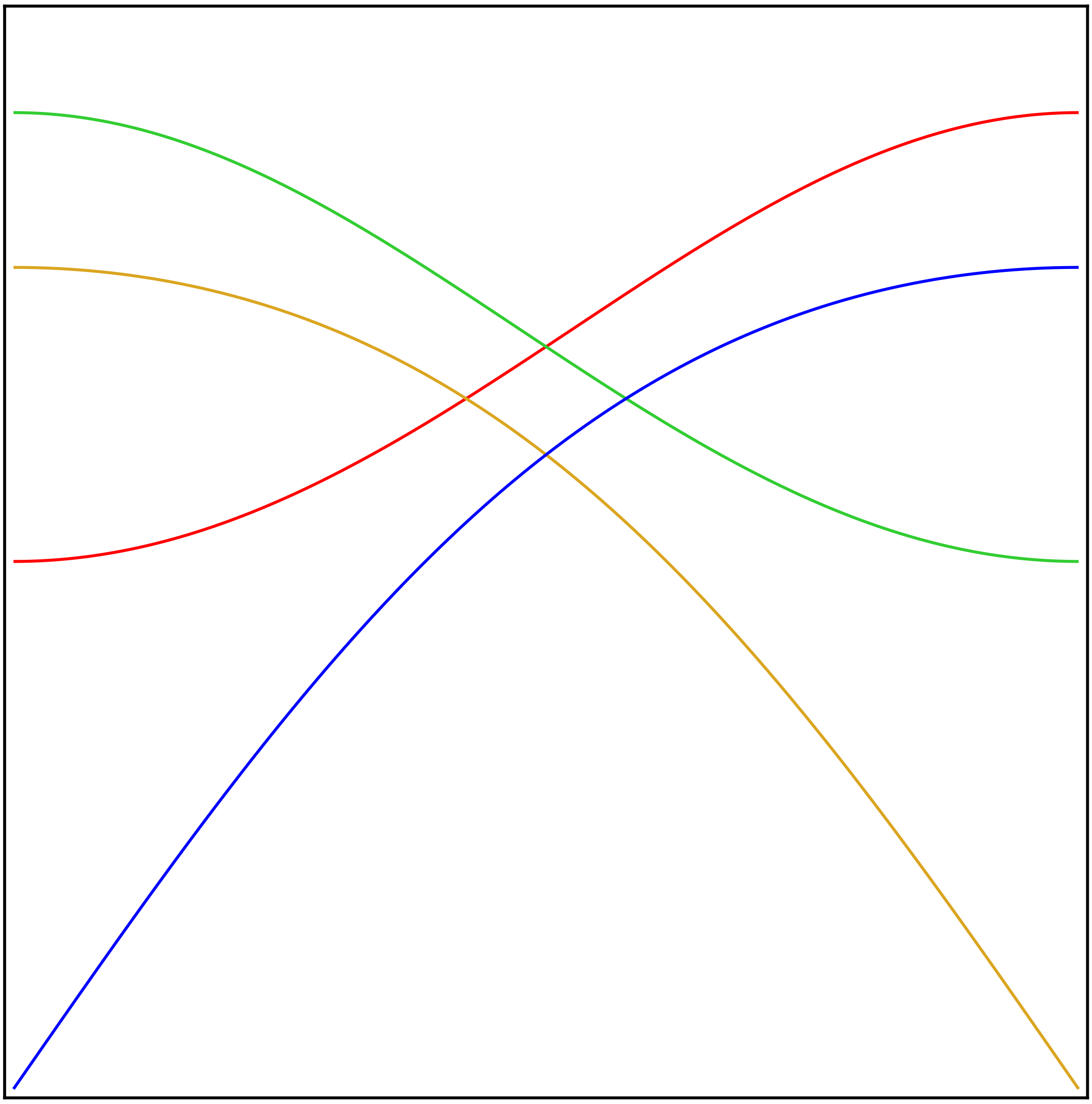
$$\cos z = \beta.$$

(e) Deduce that if $x$ takes the complex value

$$x = \tfrac{1}{4}\pi + i\log\left[\frac{\pi}{2\sqrt{2}} + \sqrt{\frac{\pi^2}{8} - 1}\right]$$

then

$$\cos(\sin x) = \sin(\cos x).$$



## SECTION B

2. The *Bessel* function, $J_0(x)$, is infinitely differentiable, and satisfies $J_0(0) = 1$, $J_0'(0) = 0$ and for all values of $x$

$$x^2 J_0'' + x J_0' + x^2 J_0 = 0.$$

By differentiating this equation $n$ times, obtain an expression for $J_0^{(n)}(0)$ in terms of lower derivatives. Hence obtain the expansion

$$J_0(x) = 1 - \tfrac{1}{4}x^2 + \tfrac{1}{64}x^4 + \dots$$

and find an expression for the general term in this series.

Use the ratio test to determine the radius of convergence of the infinite series.

Show that

$$J_0(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}.$$

3.

(a) Define what it means for a function $f(x)$ to be differentiable at $x = a$.

Using this definition, prove that if $f$ and $g$ are differentiable, then

$$(fg)' = fg' + f'g.$$

(b) Use the chain rule and the fundamental theorem of calculus to evaluate

$$\frac{d}{dx} \int_1^{x^2} f(t)\, dt.$$

Find also

$$\frac{d}{dx} \int_{x^3}^{x^2} f(t)\, dt.$$

Verify your answer for the case when $f(x) = \log x$ by evaluating the integral directly.

**Solutions**

1.(a) At $x = 0$, $\sin(\sin x) = 0$, $\cos(\sin x) = 1$, $\sin(\cos x) = \sin 1$ and $\cos(\cos x) = \cos 1$. As $1 > \pi/4$ we know $\sin 1 > \cos 1$ and so this determines which curve is which. (There are other ways, of doing this of course.)

$\sin(\cos x) = 0$ when $x = \frac{1}{2}\pi$. $\cos x = \sin x$ when $x = \pi/4$, and the other values follow.

[Identifying respectively curves and intersections **[2 and 2 marks]**

(b)

$$\cos(\sin x) = 1 - \tfrac{1}{2}(\sin^2 x) + \tfrac{1}{24}\sin^4 x = 1 - \tfrac{1}{2}(x - \tfrac{1}{6}x^3 + \ldots)^2 + \tfrac{1}{24}(x^4 + \ldots) = 1 - \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 + O(x^6)$$

**[3 marks]**

(c) If $\cos x = \cos\alpha$, then $x = 2n\pi \pm \alpha$ **[1 marks]**
If $\cos(\sin x) = \sin(\cos x) = \cos(\frac{1}{2}\pi - \cos x)$ then we have $\frac{1}{2}\pi - \cos x = 2n\pi \pm \sin x$.
Thus $\cos x \pm \sin x = \frac{1}{2}\pi - 2n\pi$ (or similar). **[2 marks]**

Now $\cos(x \pm \frac{1}{4}\pi) = \cos x \cos\frac{1}{4}\pi \mp \sin x \sin\frac{1}{4}\pi = \frac{1}{\sqrt{2}}(\cos x \mp \sin x)$ Hence (as $n$ is any integer we can replace n by $-n$ without loss of generality)

$$\cos(x \pm \tfrac{1}{4}\pi) = \frac{(4n+1)\pi}{2\sqrt{2}}$$ **[2 marks]**

as required.

Now $\pi > 3$ and so $\pi^2 > 8$ and $|\pi/2\sqrt{2}| > 1$. Furthermore $|4n+1| \geqslant 1$. We conclude there are no real values of $x$ satisfying $\cos(\sin x) = \sin(\cos x)$. Which agrees with the two curves not intersecting in part (a). **[2 marks]**

(d) If $\cos z = \beta$, then writing $\zeta = e^{iz}$,

$$2\beta = e^{iz} + e^{-iz} \qquad \Longrightarrow \qquad \zeta^2 - 2\beta\zeta + 1 = 0 \qquad \Longrightarrow \qquad \zeta = \beta \pm \sqrt{\beta^2 - 1}$$

Thus

$$iz = \log\zeta = \pm\log\left(\beta + \sqrt{\beta^2 - 1}\right) + 2n\pi i$$

$$\text{so that} \qquad z = 2n\pi \pm i\log\left(\beta + \sqrt{\beta^2 - 1}\right)$$ **[5 marks]**

(e) Combining the above, putting $\beta = \pi/2\sqrt{2}$ we get as one of our solutions

$$x = \tfrac{1}{4}\pi + i\log\left(\frac{\pi}{2\sqrt{2}} + \sqrt{\frac{\pi^2}{8} - 1}\right)$$ **[1 marks]**

**2.** Using Leibniz' rule,

$$x^2J_0^{(n+2)}+2nxJ_0^{(n+1)}+\tfrac{1}{2}n(n-1)2J_0^{(n)}+xJ_0^{(n+1)}+nJ_0^{(n)}+x^2J_0^{(n)}+2nxJ_0^{(n-1)}+\tfrac{1}{2}n(n-1)2J_0^{(n-2)}=0.$$

**[3 marks]**

So that when $x = 0$,

$$n^2J_0^{(n)}(0) = -n(n-1)J_0^{(n-2)}(0).$$ **[2 marks]**

Thus if $n$ is odd, $J_0^{(n)}(0) = 0$, **[2 marks]**

while $J_0''(0) = -\frac{1}{2}$, $J_0^{(4)} = -\frac{3}{4}J_0''(0) = \frac{3}{8}$ and in general

$$\frac{J_0^{(2k)}(0)}{(2k)!} = \frac{1}{(2k)^2}\frac{J_0^{(2k-2)}(0)}{(2k-2)!}.$$

Thus

$$J_0(x) = 1 - \tfrac{1}{4}x^2 + \tfrac{1}{64}x^4 + \dots$$ **[3 marks]**

The general term of this series is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{(2^2)(4^2)(6^2)\dots(2k)^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{2^{2k}(k!)^2}.$$ **[5 marks]**

(Only one form required.) The ratio test examines the ratio of adjacent terms

$$\left|\frac{x^nJ_0^{(n)}(0)/n!}{x^{n+2}J_0^{(n+2)}(0)/(n+2)!}\right| = \left|\frac{(n+2)^2}{x^2}\right|.$$

As $n \to \infty$, this is infinite no matter how big $x$ is. We conclude that the series converges for all $x$, i.e. the radius of convergence is infinite. **[4 marks]**
Substituting in the above formula gives $J_0(2)$ as required. (This was included as a hint for the power series. **[1 marks]**

**3.** If $f(x)$ is differentiable at $x = a$, if and only if the following limit exists:

$$\lim_{\varepsilon \to 0} \left[ \frac{f(a+\varepsilon) - f(a)}{\varepsilon} \right] \equiv f'(a). \qquad \textbf{[2 marks]}$$

Then

$$\begin{aligned}
(fg)' &= \lim_{\varepsilon \to 0} \left[ \frac{f(x+\varepsilon)g(x+\varepsilon) - f(x)g(x)}{\varepsilon} \right] \\
&= \lim_{\varepsilon \to 0} \left[ \frac{f(x+\varepsilon)g(x+\varepsilon) - f(x+\varepsilon)g(x) + f(x+\varepsilon)g(x) - f(x)g(x)}{\varepsilon} \right] \\
&= \lim_{\varepsilon \to 0} \left[ \frac{f(x+\varepsilon)g(x+\varepsilon) - f(x+\varepsilon)g(x)}{\varepsilon} \right] + \lim_{\varepsilon \to 0} \left[ \frac{f(x+\varepsilon)g(x) - f(x)g(x)}{\varepsilon} \right] \\
&= f(x) \lim_{\varepsilon \to 0} \left[ \frac{g(x+\varepsilon) - g(x)}{\varepsilon} \right] + g(x) \lim_{\varepsilon \to 0} \left[ \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \right] = fg' + gf' \qquad \textbf{[6 marks]}
\end{aligned}$$

(b) If we write $u = x^2$, then

$$\frac{d}{dx} \int_1^{x^2} f(t)\, dt = \frac{du}{dx} \frac{d}{du} \int_1^{u} f(t)\, dt = 2xf(u) = 2xf(x^2) \qquad \textbf{[3 marks]}$$

Now

$$\frac{d}{dx} \int_{x^3}^{x^2} f(t)\, dt = \frac{d}{dx} \left[ \int_1^{x^2} f(t)\, dt - \int_1^{x^3} f(t)\, dt \right] = 2xf(x^2) - 3x^2 f(x^3). \qquad \textbf{[4 marks]}$$

If $f(x) = \log x$, then this simplifies to

$$2x \log(x^2) - 3x^2 \log(x^3) = (4x - 9x^2) \log x. \qquad \textbf{[2 marks]}$$

Proceeding directly, we have, integrating by parts

$$\int_{x^3}^{x^2} \log t\, dt = \Big[ t \log t - t \Big]_{x^3}^{x^2} = x^2 \log(x^2) - x^2 - x^3 \log(x^3) + x^3 = (2x^2 - 3x^3) \log x) + x^3 - x^2.$$

Differentiating this with respect to $x$, we have

$$(4x - 9x^2) \log x + (2x^2 - 3x^3)/x + 3x^2 - 2x = (4x - 9x^2) \log x. \qquad \textbf{[3 marks]}$$

This agrees with the earlier result.